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Solving the Sixth Painlevé Equation: Towards the Classification of all the Critical Behaviors and the Connection Formulae

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Abstract: The critical behavior of a three real parameter class of solutions of the sixth Painlevé equation is computed, and parametrized in terms of monodromy data of the associated 2×2 matrix linear Fuchsian system of ODE. The class may contain solutions with poles accumulating at the critical point. The study of this class closes a gap in the description of the transcendents in one to one correspondence with the monodromy data. These transcendents are reviewed in the paper. Some formulas that relate the monodromy data to the critical behaviors of the four real (two complex) parameter class of solutions are missing in the literature, so they are computed here. A computational procedure to write the full expansion of the four and three real parameter class of solutions is proposed.

1 Introduction

The history, importance and applications of the Painlevé equations have been widely discussed in the literature and assumed to be known (for a review, see [11]). The equation PVI is:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right], \qquad \text{(PVI)}.$$

The general solution has no movable essential singularities or branch points, which are possibly located only at the *critical points* $x = 0, 1, \infty$. The behavior of a solution when $x \to 0, 1, \infty$, is called *critical behavior*. The other movable singularities are poles. The absence of movable critical points means that a solution can be meromorphically extended to the universal covering of a punctured complex sphere, determined only by the equation. Thus PVI shares a fundamental property of the linear equations defining classical transcendental functions.

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Following the review [11], an expression is called *explicit* when it is given in terms of a finite algebraic combination of elementary and elliptic functions, and a finite number of contour integrals (and quadratures) of these functions. Classical linear special functions admit explicit representations. The general solution of a Painlevé equation does not, as it is proved by H.Umemura in [30]. Therefore, it is not a classical function. It is a new function, called a *Painlevé transcendent*.[2]

Following [11] (page 8), *solving* PVI means: i) Determine the *explicit* critical behavior of the transcendents at the critical points. Such a behavior must be given by an explicit formula in terms of two integration constants. ii) Solve the *connection problem*, namely: find the *explicit* relations among couples of integration constants at different critical points. The above i) and ii) are the problem of *global analysis* of the equation. Solution of i) and ii) means that a Painlevé transcendents can be efficiently used in applications as it is the case for special functions. It was thought that the global analysis is possible only for linear equations, namely only for classical linear special functions. But the *method of monodromy preserving deformations* has made the global analysis possible also for Painlevé equations.[3]

The critical behaviors for a two complex (four real) parameter class of solutions were computed and parametrized in terms of monodromy data of an associated Fuchsian system of ODEs, by Jimbo in [21]. Jimbo's paper is the foundation of all the works on PVI based on the method of monodromy preserving deformations which have followed. Some authors have determined critical behaviors not included in Jimbo's class, with different methods. Among them, the works of S.Shimomura (the results are summarized in [20]) and A.D.Bruno, I.V. Goryuchkina ([2] [3] [4] [5] [6]) are local approaches, which do not determine the connection formulae, but essentially determine all the critical behaviors or asymptotic expansions. In [14] [15] [16] [17], in the framework of the method of

[2]H. Umemura proved the of irreducibility of the Painlevé equations [30] [31] [32]. The term "explicit" expression is equivalent to the notion of classical function. Following [30], a function is called *classical* if it is given in terms of a finite iteration of *permissible operations* applied to rational functions. They are the derivation, rational combination (sum, product, quotient), algebraic combinations (the expression is a root of a polynomial whose coefficients are rational functions (and then, after iteration, classical functions)), contour integrals and quadratures, solution of a linear homogeneous differential equation whose coefficients are rational functions (or classical functions, after iteration), a solution of an algebraic differential equation of the first order whose coefficients are rational functions (or classical functions), composition with abelian functions (the expression is $\varphi(f_1(x),...,f_n(x))$, where f_1,...,f_n are rational or classical functions, and $\varphi:\mathbf{C}^n/\Gamma\to\mathbf{C}$ is meromorphic, Γ is a lattice). The reader may note that the elementary transcendental functions are classical functions (they are the algebraic functions, or a function which is obtained from an algebraic function by integration (like the exponential, the trigonometric and hyperbolic functions), or the inverse of such an integral (like the logarithm, the elliptic functions, etc)). Umemura proved in [30] that the general solution of a Painlevé equation is not a classical function. H.Watanabe [33] applied the argument to PVI, and showed that a solution of PVI is either algebraic, or solves a Riccati equation (one-parameter family of classical solutions), or it is not a classical function. All the algebraic solutions were classified in [9] when $\beta=\gamma=0$, $\delta=\frac{1}{2}$, and then in [24] for the general PVI.

[3]A more restrictive definition of "solving" should include the distribution of the poles (movable singularities) of the transcendents. This problem for PVI is still open (in [14], the behavior on the universal covering of a critical point is analyzed and it is shown that if the poles exist, they are distributed in spirals converging to the critical point).

monodromy preserving deformations, critical behaviors not included in Jimbo's class are constructed and parametrized in terms of associated monodromy data. Accordingly, the transcendents can be cassified into a few classes (one beeing that of Jimbo's), depending on their local behavior and their correspondence with sub spaces of the space of the associated monodromy data. This fact is reviewed below, in subsection 1.1. The last class (a three real parameter cass of solutions) has not been studied yet, and it is studied here. Its critical behavior and parametrization in terms of monodromy data is given in the present paper. Together with it, in the paper a complete review of the parametrization of critical behaviors in terms of monodromy data is given for all the four and three real parameter solutions. Some formulas missing in the literature are computed.

According to the above definition of "solving", the paper by Jimbo [21] and [14] [15] [16] [17], together with the present paper, "solve" PVI. Namely, the critical behaviors and the parametrization in terms of monodromy data have been found for of all the transcendents that are in one-to-one correspondence with points in the space of the associated monodromy data.

Before stating the results of the paper, we give a review of the critical behaviors at $x = 0$.

1.1 A Review of Critical Behaviors

In the following, let $|\arg x| < \pi$, $|\arg(1-x)| < \pi$, so that all functions of x will be understood as x-branches. According to [22], PVI is the condition of isomonodromy deformation for a 2×2 fuchsian system with four singularities $0, x, 1, \infty$:

$$\frac{d\Psi}{d\lambda} = A(x,\lambda)\ \Psi, \qquad A(x,\lambda) := \left[\frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda - x} + \frac{A_1(x)}{\lambda - 1}\right], \quad \lambda \in \mathbf{C}. \tag{1}$$

The traces of the matrices are zero, and the eigenvalues are fixed by PVI. This facts are reviewed in Section 2. A fundamental solution Ψ has branch points in $\lambda = 0, x, 1$. Fix a base point and a base of loops Γ like in figure 1. When λ goes around a small loop around a branch point, the fundamental solution transforms like $\Psi \mapsto \Psi M_i$, $i = 0, x, 1$. The 2×2 matrices M_0, M_x, M_1 are called the monodromy matrices of the fundamental solution.

Given PVI (namely, given α, β, γ and δ), there is a one-to-one correspondence between a triple of monodromy martices, associated to the base of loops Γ, and a branch of a PVI transcendent. This happens in the generic case (which will be made precise in Section 2). A branch is uniquely identified by the monodromy data associated to the basis of loops Γ:

$$y(x) = y(x; \mathrm{Tr}M_0,\ \mathrm{Tr}M_x,\ \mathrm{Tr}M_1,\ \mathrm{Tr}M_0M_x,\ \mathrm{Tr}M_xM_1,\ \mathrm{Tr}M_0M_1)$$

This will be precisely explained in Section 2. Here it is enough to understand that the critical behavior at a critical point depends on two integration constants (which in general are 4 real parameters, but in sub cases they may reduce to 3 or 2 real parameters). The parametrization of the integration constants in terms of monodromy data uniquely identifies the branch of the transcendent. The expicit parametrization will be given in Section 5, for the 4-real parameter and 3-real parameter branches.

What kind of critical behaviors we may expect at $x = 0, 1, \infty$ depends on the values of $\mathrm{Tr}M_0M_x$, $\mathrm{Tr}M_xM_1$, $\mathrm{Tr}M_0M_1$ respectively. For example, the type of behavior at $x = 0$ (for example, a two real parameter solution with logarithmic behavior, or a solution of Jimbo's, etc) is decided by the value of $\mathrm{Tr}(M_0M_x)$.

Here the classes of critical behaviors are reviewed, corresponding to monodromy groups which have the property of being irreducible, and such that they are in one to one correspondence with branches of PVI transcendents (namely, none of the monodromy matrices $M_0, M_x, M_1, M_1M_0M_x$ is the identity).

$\diamond$) [**Small power type behaviors (Jimbo). 4-real parameters**:] M.Jimbo was the first to determine the critical behaviors for a wide class of transcendents. In [21] he proved that PVI admits solutions with branches behaving as follows:

$$y(x) = \begin{cases} a_0 x^{1-\sigma_0}(1+\delta_0(x)), \quad x \to 0 \\ \\ 1 - a_1(1-x)^{1-\sigma_1}(1+\delta_1(1-x)), \quad x \to 1 \\ \\ a_\infty x^{\sigma_\infty}(1+\delta_\infty(x^{-1})), \quad x \to \infty \end{cases} \tag{2}$$

where $a_i, \sigma_i \in \mathbf{C}$ are integration constants such that:

$$a_i \neq 0, \qquad 0 < \Re\sigma_i < 1.$$

$\delta_i(\zeta)$ are higher order terms, $\delta(\zeta) = O(\max\{|\zeta|^{\Re\sigma}, |\zeta|^{1-\Re\sigma}\})$. Jimbo determined the parametrization of the couples (a_0, σ_0) (a_1, σ_1), $(a_\infty, \sigma_\infty)$ in terms of monodromy data. *The parametrization identifies the specific branch.* In particular he proved that:

$$2\cos(\pi\sigma_0) = \mathrm{Tr}(M_0M_x), \quad 2\cos(\pi\sigma_1) = \mathrm{Tr}(M_xM_1), \quad 2\cos(\pi\sigma_\infty) = \mathrm{Tr}(M_0M_1) \tag{3}$$

The restriction on $\Re\sigma_i$ means that the solutions correspond to the following subspace of the space of monodromy matrices:

$$\mathrm{Tr}(M_iM_j) \notin (-\infty, -2] \cup [2, \infty).$$

For special values of σ the above behaviors are modified. For example, for $x \to 0$, one has (see [16], plus section 7 and section 8.1.1 of the present paper):

$$y(x) = \frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+\sqrt{1-2\delta}}x \mp \frac{r}{\sqrt{-2\beta}+\sqrt{1-2\delta}}\, x^{1+\sigma} + O(x^2), \quad \sigma = \pm(\sqrt{-2\beta}+\sqrt{1-2\delta}),$$

$$y(x) = \frac{\sqrt{-2\beta}}{\sqrt{-2\beta}-\sqrt{1-2\delta}}x \mp \frac{r}{\sqrt{-2\beta}-\sqrt{1-2\delta}}\, x^{1+\sigma} + O(x^2), \quad \sigma = \pm(\sqrt{-2\beta}-\sqrt{1-2\delta}),$$

Here $r \in \mathbf{C}$ is the integration constant and the condition $-1 < \Re\sigma < 1$ must hold.

Not only in Jimbo's case, but in general, the critical behavior of $y(x)$ is decided by three constants σ_0, σ_1, σ_∞, determined by (3) plus the conditions $0 \leq \Re\sigma_i \leq 1$.

Below, behaviors are given only for $x \to 0$ ($\arg(x)$ bounded). We denote $\sigma := \sigma_0$. The other critical points $x = 1, \infty$ will be described in the paper.

$\diamond$) [**Sine-type oscillatory behaviors. 3 real parameters**:] If $\Re\sigma = 0$, the critical behavior follows from Jimbo's results (see Appendix I) and the equivalent method of [16]. There exist a transcendent with a branch at $x = 0$ behaving as follows:

$$y(x) = x\left\{iA\sin\left(i\sigma\ln x + \phi\right) \ + B + \delta^*(x)\right\}, \quad \delta^*(x) = O(x), \quad x \to 0 \tag{4}$$

$$\sigma,\ \phi \text{ integration constants.} \quad B = \frac{\sigma^2 - 2\beta - 1 + 2\delta}{2\sigma^2}, \quad A^2 + B^2 = -\frac{2\beta}{\sigma^2}.$$

In this case:

$$2\cos\pi\sigma = \mathrm{Tr}(M_0 M_x) > 2.$$

The parametrization of σ and ϕ in terms of monodromy data uniquely identifies the branch.

$\diamond$) [**Log-type behaviors. 2 real parametes**:] If $\sigma = 0, 1$, namely:

$$\mathrm{Tr}(M_0 M_x) = \pm 2,$$

There are transcendents with logarithmic branches (see [21] formula $(1.9)'$, and [16] [17]). In [16] [17], the branches are written as follows. When $\sigma = 0$, $\mathrm{Tr}(M_0 M_x) = 2$:

$$y(x) = x\left[\frac{1 + 2\beta - 2\delta}{4}\left(\ln x + \frac{4r + 2\sqrt{-2\beta}}{2\delta - 2\beta - 1}\right)^2 + \frac{2\beta}{2\beta + 1 - 2\delta}\right] + O(x^2\ln^3 x), \quad 2\beta \neq 2\delta - 1;$$

$$y(x) = x(r \pm \sqrt{-2\beta}\ln x) + O(x^2\ln^2 x), \quad 2\beta = 2\delta - 1.$$

For the second solution, the subgroup $< M_0, M_x >$ is reducible. r is the integration constant.

When $\sigma = 1$, $\mathrm{Tr}(M_0 M_x) = -2$:

$$y(x) = \frac{2}{(\gamma - \alpha)\ln^2 x}\left[1 + \frac{4r + \sqrt{8\alpha}}{\gamma - \alpha}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \quad \alpha \neq \gamma;$$

$$y(x) = \frac{1}{\pm\sqrt{2\alpha}\ \ln x}\left[1 \mp \frac{r}{\sqrt{2\alpha}\ \ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \quad \alpha = \gamma.$$

For the second solution, the subgroup $< M_0 M_x, M_1 >$ is reducible. r is the integration constant. Its parametrized in terms of monodromy data is in [17]. This identifies the branch.

$\diamond$) [**Taylor expansions. 2 real parameters**:] Solutions with branches which admit a Taylor expansions at a critical point are studied in [16], [23]. According to [16], such expansions (which are convergent for small $|x|$ by the argument of [23]) are the following 1), 2), 3) below:

1) Degenerate solutions $y = 0, x, 1$.

2) The *Basic Expansions* i), ii), iii) below:.

i) When $\alpha \neq 0$ and $\sqrt{2\alpha} \pm \sqrt{2\gamma}$ is not integer:

$$y(x) = \frac{\sqrt{\alpha} \pm \sqrt{\gamma}}{\sqrt{\alpha}} \mp \frac{\sqrt{\gamma}\left[(\sqrt{2\alpha} \pm \sqrt{2\gamma})^2 - 2\delta + 2\beta\right]}{2\sqrt{\alpha}((\sqrt{2\alpha} \pm \sqrt{2\gamma})^2 - 1)}\ x + \sum_{n=2}^{\infty} c_n(\sqrt{\alpha}, \pm\sqrt{\gamma}, \beta, \delta)\ x^n$$

$$\mathrm{Tr}(M_0 M_x) = -2\cos\pi(\sqrt{2\alpha} \pm \sqrt{2\gamma}).$$

ii) When $\alpha \neq 0$, but $\sqrt{2\alpha} \pm \sqrt{2\gamma} = 1$ and $1 - 2\delta + 2\beta = 0$:

$$y(x) = \pm\frac{1}{\sqrt{2\alpha}} + rx + \sum_{n=2}^{\infty} c_n(r; \sqrt{\alpha}, \beta)\ x^n, \qquad r \in \mathbf{C}$$

$$\mathrm{Tr}(M_0 M_x) = 2.$$

iii) When $\alpha = 0$ and $\sqrt{\alpha} \pm \sqrt{\gamma} = 0$:

$$y(x) = r + (1-r)(\delta - \beta)x + \sum_{n=2}^{\infty} c_n(a; \beta, \delta)\ x^n, \qquad r \in \mathbf{C}$$

$$\mathrm{Tr}(M_0 M_x) = -2.$$

In all i), ii), iii) above, the subgroup $< M_0 M_x, M_1 >$ is reducible. The square roots $\sqrt{\alpha}$, $\sqrt{\gamma}$ have arbitrary sign. The coefficients are rational functions of their arguments. The parametrization of r in terms of monodromy data is in [16]. It uniquely identifies the branch.

3) All the expansions obtained from 2) by the birational transformations of PVI that do not change x. For example, the bitrational transformation (30) gives:

i) When $\beta \neq 0$ and $\sqrt{-2\beta} \pm \sqrt{1-2\delta} \neq 0$:

$$y(x) = \frac{\sqrt{-2\beta}\ x}{\sqrt{-2\beta} \pm \sqrt{1-2\delta}} \pm \frac{\sqrt{-2\beta}\sqrt{1-2\delta}\left[(\sqrt{-2\beta} \pm \sqrt{1-2\delta})^2 + 2\gamma - 2\alpha - 1\right]\ x^2}{2(\sqrt{-2\beta} \pm \sqrt{1-2\delta})^2\left[(\sqrt{-2\beta} \pm \sqrt{1-2\delta})^2 - 1\right]} +$$

$$+ \sum_{n=3}^{\infty} b_n(\alpha, \sqrt{\beta}, \sqrt{1-2\delta}, \gamma) x^n$$

$$\mathrm{Tr}(M_0 M_x) = -2\cos\pi(\sqrt{-2\beta} \pm \sqrt{1-2\delta}).$$

ii) When $\beta \neq 0$ but $(\sqrt{-2\beta} \pm \sqrt{1-2\delta})^2 = 1$ and $\alpha = \gamma$:

$$y(x) = \pm\sqrt{-2\beta}\ x\ + r\ x^2\ + \sum_{n=3}^{\infty} b_n(r; \sqrt{\alpha}, \sqrt{\beta}) x^n, \qquad r \in \mathbf{C}$$

$$\mathrm{Tr}(M_0 M_x) = -2.$$

iii) When $\beta = 1 - 2\delta = 0$:

$$y(x) = rx + \frac{r(r-1)}{2}(2\gamma - 2\alpha - 1)x^2 + \sum_{n=3}^{\infty} b_n(r;\alpha,\gamma)x^n, \qquad r \in \mathbf{C}$$

$$\mathrm{Tr}(M_0 M_x) = 2.$$

In all I), II), III), the subgroup $< M_0, M_x >$ is reducible.

$\diamond$) [**Inverse sine-type oscillatory behaviors. 3 real parameters**:] The above results "solve" PVI for all the values of $\mathrm{Tr}(M_i M_j)$, except for the case $\mathrm{Tr}(M_i M_j) < -2$, namely the case when $\Re\sigma_i = 1$. This case is studied in the present paper. The result, at $x = 0$, is Proposition 1: there exist transcendents with a branch at $x = 0$ having the following behavior:

$$y(x) = \frac{1}{iA\sin\Big(i(1-\sigma_0)\ln x + \phi_0\Big) + B + \delta_0^*(x)}, \qquad \delta_0^*(x) = O(x), \quad x \to 0 \tag{5}$$

$$\sigma_0,\ \phi_0 \text{ integration constants.} \quad B = \frac{\Im\sigma_0^2 + 2\gamma - 2\alpha}{2\Im\sigma_0^2}, \quad A^2 + B^2 = -\frac{2\alpha}{(\Im\sigma_0)^2}$$

In this case:

$$2\cos(\pi\sigma_0) = \mathrm{Tr}(M_0 M_x) < -2, \qquad \Re\sigma_0 = 1.$$

The parametrization of σ and ϕ in terms of monodromy data uniquely identifies the branch.

A similar classification holds at $x = 1, \infty$. Note that a solution with a behavior falling in one class at a critical point, may have a behavior of a different type at another critical point, depending on the values of $\mathrm{Tr}(M_i M_j)$.

1.2 Results of the Paper

The relevant results of this paper are the following three points.

1) In this paper PVI is solved in the missing case $\mathrm{Tr}(M_j M_k) < -2$, $\Re\sigma_i = 1$. Precisely:

– The critical behaviors when $x \to 0, 1, \infty$, with $\arg x$ and $\arg(1-x)$ bounded is computed. Let PVI be given, and let the monodromy data be given (such that the one-to-one correspondence holds true). Let $x \to 0$ inside a sector. Let $2\cos\pi\sigma_0 = \mathrm{Tr}(M_0 M_x) < -2$, $\Re\sigma_0 = 1$. The solution corresponding to these monodromy data has the critical behavior (5) [**Inverse sine-type oscillatory behaviors**, Proposition 1]. Observe that:

$$\sin\Big(i(1-\sigma_0)\ln x + \phi\Big) = \sin\Big(\Im\sigma_0 \ln x + \phi\Big)$$

gives a purely oscillating contribution when $x \to 0_+$. The above behavior also predicts the occurrence of poles close to $x = 0$, when the denominator vanishes. This is the reason why the correction $\delta_0^*(x)$ in the denominator must be kept. Namely, one cannot write $y(x) = \{iA\sin\Big(i(1-\sigma_0)\ln x + \phi_0\Big) + B\}^{-1}(1 + O(x))$, because this would affect

the position of the poles. An example which makes this point clear is the Picard-type solution (see Appendix II, solution (68)):

$$y(x)=\frac{1+O(x)}{\sin^2\left(\frac{\Im\sigma}{2}\ln x+\phi+\frac{\Im\sigma}{2}\frac{F_1(x)}{F(x)}\right)}+O(x),\quad x\to 0$$

where $F(x)$ and $F_1(x)$ are the hypergeometric-like functions (57) and (58). The poles close to $x=0$ are determined by the solutions of $\frac{\Im\sigma}{2}\ln x+\phi+\frac{\Im\sigma}{2}\frac{F_1(x)}{F(x)}=k\pi$, $k\in\mathbf{Z}$, which lie in a neighborhood of $x=0$. The distribution of poles in the general case will be studied in another paper.

– Connection Problem. The parametrization of the critical behavior in terms of monodromy data is given. In section 5, Proposition 6 the critical behaviors at the three critical points $x=0,1,\infty$ is given, when $\Re\sigma_i=1$, $i=0,1,\infty$. They are as follows:

$$y(x)=1-\frac{1}{iA_1\sin\Big(i(1-\sigma_1)\ln(1-x)+\phi_1\Big)+B_1+\delta_1^*},\quad \delta_1^*=O(1-x),\quad x\to 1$$

$$y(x)=\frac{x}{iA_\infty\sin\Big(i(\sigma_\infty-1)\ln x+\phi_\infty\Big)+B_\infty+\delta_\infty^*(x)},\quad \delta_\infty^*=O\left(\frac{1}{x}\right),\quad x\to\infty$$

The coefficients $A_1,A_\infty,B_1,B_\infty$ are given in terms of $\alpha,\beta,\gamma,\delta$ in Proposition 6. In Proposition 6 the integration constants $\phi_0,\phi_1,\phi_\infty$ are also given as functions of the coefficients of PVI and of the monodromy data $\mathrm{Tr}(M_jM_k)$ (see (45)). This parametrization fixes the branches. Conversely, in Proposition 7, the formulae which express $\mathrm{Tr}(M_jM_k)$ as functions of the coefficients of PVI and of the integration constants are given. See formulae (46). In this way, one is able to compute any of the couples (σ_0,ϕ_0), (σ_1,ϕ_1), $(\sigma_\infty,\phi_\infty)$ as a function of another. This *solves the connection problem.*

The author already studied the case $\mathrm{Tr}(M_iM_j)<-2$ in [15], [14], with the elliptic representation. But the critical behavior obtained was $y=\{\sin^2(\frac{\Im\sigma}{2}\ln x+\psi(x))+O(x)\}^{-1}$, where $\psi(x)=\sum_{n\geq 0}\psi_n x^{-in\Im\sigma}$ is an oscillatory function. The same behavior follows from the results of Shimomura ([20], chapter 4, section 2). Unfortunately, the function $\psi(x)$ in the sine makes the formula uncomputable. The meaning of the result of the present paper is that $\psi(x)$ *has been brought out* of the sin(..) and computed. The behavior of [15], [14], [20] of course must coincide with that of the present paper. This is possible because one can always write $iA\sin(\nu\ln x+\phi)+B$ (where $\nu\in\mathbf{R}$, $\phi\in\mathbf{C}$) as $\sin^2(\frac{\nu}{2}\ln x+\sum_{n\geq 0}\psi(x))$, where $\psi(x)$ is an oscillating function (not vanishing for $x\to 0$) computable in an elementary way. If $\psi(x)$ can be expanded in series in a suitable domain, then the series turns out to be necessarily of the form $\psi(x)=\sum_{n\geq 0}\psi_n x^{-in\nu}$, $\psi_n\in\mathbf{C}$. See Appendix II, subsection 9.3, for the details.

It is to be cited the paper [6], where all the asymptotic expansions are obtained with a power geometric technique [7]. This technique does not allow to solve the connection problem. In [6], formula (7), one finds an expansion that, in the notation of the present paper, becomes $y(x)=1/[iA\sin(i(1-\sigma)\ln x+\phi)+B]+\sum_{\Re s\geq 1}c_s x^s$. The absence of a term $\delta^*(x)=O(x)$ in the denominator, which is essential to determine the position of

the poles, means that the expansion of [6] gives the asymptotics when $x \to 0$ far from the poles.

2) In this paper, section 7, the recursive procedure is given to compute at any order the expansions, for $x \to$ critical point, of the 4 and 3 real parameters solutions (namely, solutions such that $\mathrm{Tr}(M_j M_k) \neq \pm 2$, $0 \leq \Re\sigma_i \leq 1$, $\sigma_i \neq 0, 1$). The ordering of the terms in the expansion is sensibly depending on the initial conditions (i.e. on the exponent of the leading term). For this reason, so far it has been thought that the expansion is formally uncomputable in general. It is shown that this is not the case. The procedure to compute it in general is given, independently on the initial conditions (i.e. the value of σ_i). The convergent expansions for $x \to 0$ are:

$$\delta(x) = \sum_{n=0}^{\infty} x^n \sum_{m=-n}^{n+2} \tilde{c}_{nm}(\sigma, a, \alpha, \beta, \gamma, \delta) x^{m\sigma} - 1 = O(\max\{x^{\Re\sigma}, x^{1-\Re\sigma}\}), \quad \tilde{c}_{00} = 1$$

for the solution (2).

$$\delta^*(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n-1}^{n+1} b_{nm}(\sigma, \phi, \alpha, \beta, \gamma, \delta) x^{m\sigma} = O(x)$$

for the solution (4).

$$\delta^*(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n-1}^{n+1} d_{nm}(\sigma, \phi, \alpha, \beta, \gamma, \delta)\ x^{m(1-\sigma)} = O(x)$$

for the solution (5). The procedure is given to compute the coefficients $\tilde{c}_{nm}, b_{nm}, d_{nm}$ in section 7. They are rational functions of the integration constants σ, $\exp\{i\phi\}$ and of $\alpha, \beta, \gamma, \delta$.

3) In this paper are also computed the explicit formulae which express $\mathrm{Tr}(M_j M_k)$ as functions of the coefficients of PVI and of the integration constants for 4 and 3 real parameter solutions, namely for $0 \leq \Re\sigma_i \leq 1$, $\sigma_i \neq 0, 1$ (see (33) for $0 \leq \Re\sigma_i < 1$ and (46) for $\Re\sigma_i = 1$). This is the first time that the explicit formulae appear in the literature for the general PVI (for the special case $\beta = \gamma = 0$, $\delta = 1/2$ they are given in [9] and [14]). These formulae are necessary for the solution of the connection problem.

The relevance of this paper is that, together with all previous contributions, first of all that of Jimbo [21] and then the series of papers [14] [15] [16] [17], PVI may be considered "solved" (in the meaning stated in the introduction), solved in all the cases when there is a one-to-one correspondence between monodromy data and Painlevé transcendents and $< M_0, M_x, M_1 >$ is irreducible. This is because *all* the critical behaviors have been obtained, and *almost all* the parametrizations of the integration contants in terms of monodromy data. "Almost" means that some special values of the monodromy data θ_μ (to be introduced in section 2, see (6)) are poles of the connection formulae. It is possible to compute the formulae in these special cases as well, with no conceptual changes in the general scheme of [16] and [21]. These very time-consuming computations will be done only when one gets specifically interested in some special case. In [9] [14], all the

computations for the relevant special case of PVI associated to a Frobenius manifold are done.

This paper is organized as follows:
- Section 2: review the isomonodromy deformation approach to PVI.
- Section 3: statement of the critical behavior (5), when $x \to 0$ [Proposition 1]. Review of (2) and (4).
- Section 4: proof of (5), via a symmetry of PVI which transforms (4) into (5).
- Section 5: The connection problem. All the formulae relating monodromy data and integration constants are given for the small power type behaviors, the sine-type oscillatory behaviors and inverse sine-type oscillatory behaviors.
- Section 6: example of the above connection formulae for PVI associated to a Frobenius manifold.
- Section 7: recursive computation of the full expansion of $y(x)$ (of $\delta(x)$, $\delta^*(x)$) at the critical points.
- Appendix I: review of the procedure of Jimbo to obtain (2) and (4).
- Appendix II: review of the elliptic representation and proof of the convergence of the full expansion of $y(x)$ (of $\delta(x)$, $\delta^*(x)$).

2 Review of the Isomonodromy Deformations

(PVI) is the isomonodromy deformation equation of the 2×2 matrix linear Fuchsian system of ODEs given in equation (1). The 2×2 matrices $A_i(x)$ depend on the parameters $\alpha,\beta,\gamma,\delta$ according to the following relations:

$$A_0 + A_1 + A_x = -\frac{\theta_\infty}{2}\sigma_3, \quad \theta_\infty \neq 0,$$

$$\text{Eigenvalues }(A_i) = \pm\frac{1}{2}\theta_i, \quad i = 0,1,x;$$

$$\theta_0^2 = -2\beta, \quad \theta_x^2 = 1-2\delta, \quad \theta_1^2 = 2\gamma, \quad (\theta_\infty - 1)^2 = 2\alpha \tag{6}$$

Here $\sigma_3 := \mathrm{diag}(1,-1)$ is the Pauli matrix. The condition $\theta_\infty \neq 0$ is not restrictive, because $\theta_\infty = 0$ is equivalent to $\theta_\infty = 2$. The equations of monodromy preserving deformation (Schlesinger equations), can be written in Hamiltonian form and reduce to (PVI), being the transcendent $y(x)$ the solution λ of $A(x,\lambda)_{1,2} = 0$. Namely:

$$y(x) = \frac{x\ (A_0)_{12}}{x\ \left[(A_0)_{12} + (A_1)_{12}\right] - (A_1)_{12}}, \tag{7}$$

The matrices $A_i(x)$, $i = 0,x,1$, depend on $y(x)$, $\frac{dy(x)}{dx}$ and $\int y(x)$ through rational functions, which are given in [22]

The standard choice of a fundamental matrix Ψ is as follows:

$$\Psi(\lambda)=\begin{cases}\left[I+O\left(\frac{1}{\lambda}\right)\right]\ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, & \lambda\to\infty;\\[2mm] \psi_0(x)[I+O(\lambda)]\ \lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0}C_0, & \lambda\to 0;\\[2mm] \psi_x(x)[I+O(\lambda-x)]\ (\lambda-x)^{\frac{\theta_x}{2}\sigma_3}(\lambda-x)^{R_x}C_x, & \lambda\to x;\\[2mm] \psi_1(x)[I+O(\lambda-1)]\ (\lambda-1)^{\frac{\theta_1}{2}\sigma_3}(\lambda-1)^{R_1}C_1, & \lambda\to 1;\end{cases}\tag{8}$$

Here $\psi_0(x)$, $\psi_x(x)$, $\psi_1(x)$ are the 2×2 diagonalizing matrices of $A_0(x)$, $A_1(x)$, $A_x(x)$ respectively. They are defined by multiplication to the right by arbitrary diagonal matrices, possibly depending on x. C_ν, $\nu=\infty,0,x,1$, are invertible *connection matrices*, independent of x [22]. Each R_ν, $\nu=\infty,0,x,1$, is also independent of x, and:

$$R_\nu=0 \text{ if } \theta_\nu\notin\mathbf{Z},\qquad R_\nu=\begin{cases}\begin{pmatrix}0&*\\0&0\end{pmatrix}, & \text{if } \theta_\nu>0 \text{ integer}\\[2mm] \begin{pmatrix}0&0\\ *&0\end{pmatrix}, & \text{if } \theta_\nu<0 \text{ integer}\end{cases}$$

If $\theta_i=0$, $i=0,x,1$, then R_i is to be considered the Jordan form $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ of A_i.

Let a basis of loops in the order $(1,2,3)=(0,x,1)$ be fixed. There are several (infinite) choices of such a basis. Here the basis Γ of figure 1 is chosen (other possible simple choices are the basis Γ_0 and $\Gamma_1$in figure 2).

Let the x-plane be cut by the condition that $|\arg x|<\pi$, $|\arg(1-x)|<\pi$, so that $A(x,\lambda)$ and $y(x)$ make sense as x-branches.

When λ goes around a counter-clockwise loop around $0,x,1$, then Ψ is transformed by right multiplication by *monodromy matrices* M_0,M_x,M_1:

$$\Psi\mapsto\Psi M_j,\quad M_j=C_j^{-1}\exp\{i\pi\theta_j\sigma_3\}\exp\{2\pi i R_j\}C_j,\quad j=0,x,1.$$

For the loop γ_∞: $\lambda\mapsto\lambda e^{-2\pi i}$, $|\lambda|>\max\{1,|x|\}$, the monodromy at infinity is:

$$M_\infty=\exp\{i\pi\theta_\infty\}\ \exp\{-2\pi i R_\infty\}.$$

The following relation holds:

$$\gamma_0\gamma_x\gamma_1\gamma_\infty=1,\quad M_1M_xM_0M_\infty=I$$

The **monodromy data** of the fuchsian system, *with respect to a basis of loops* Γ, are the following set of data:

a) The exponents $\pm\theta_0,\pm\theta_x,\pm\theta_1,\pm(\theta_\infty-1)$, with $\theta_\infty\neq 0$.
b) Matrices R_0,R_x,R_1,R_∞, such that:

$$R_\nu=0 \text{ if } \theta_\nu\notin\mathbf{Z},\qquad R_\nu=\begin{cases}\begin{pmatrix}0&*\\0&0\end{pmatrix}, & \text{if } \theta_\nu>0 \text{ integer}\\[2mm] \begin{pmatrix}0&0\\ *&0\end{pmatrix}, & \text{if } \theta_\nu<0 \text{ integer}\end{cases}$$

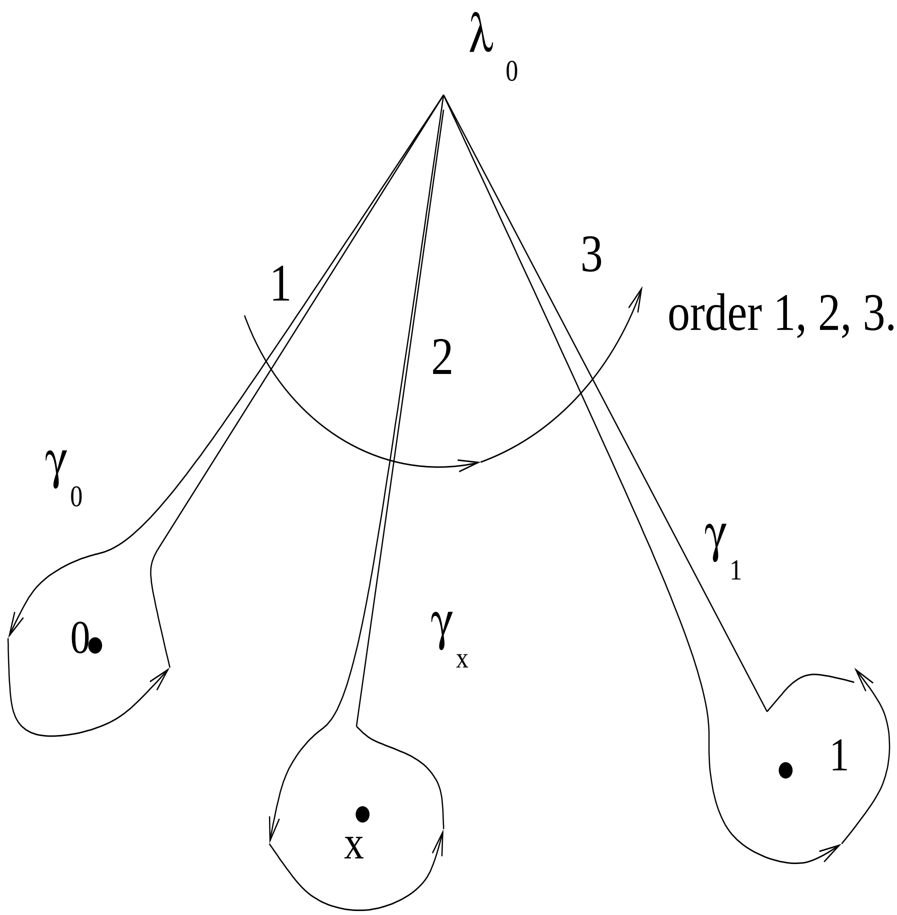

Figure 1: The ordered basis of loops Γ

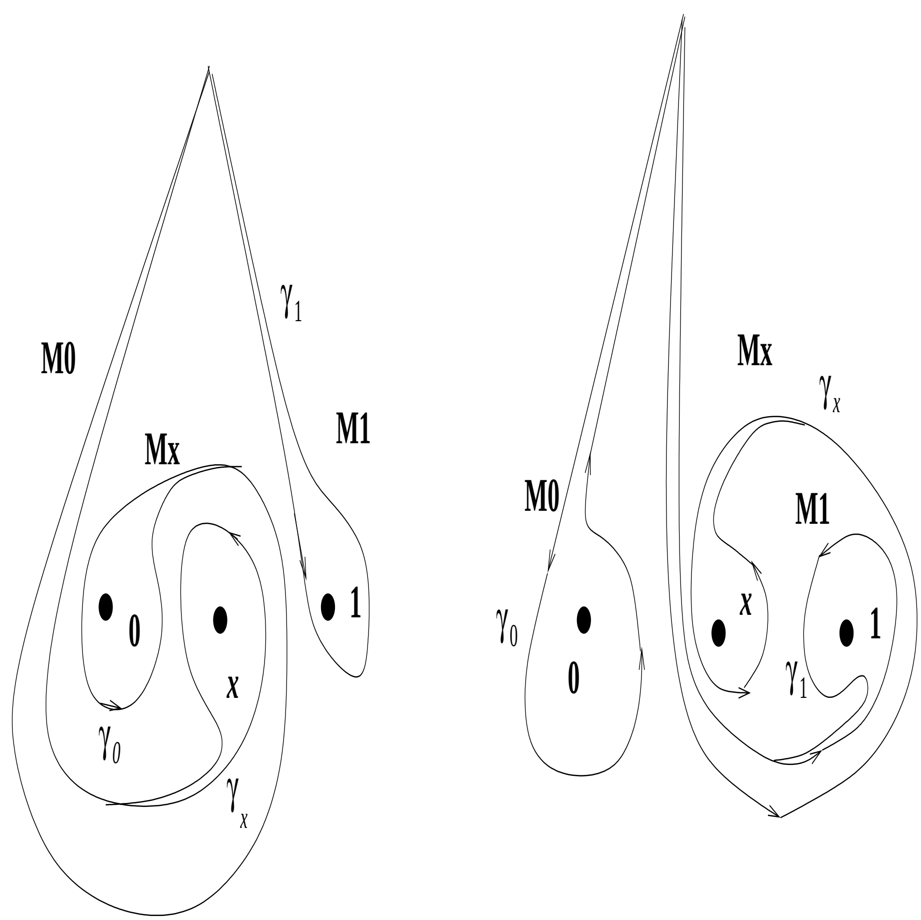

Figure 2: Other choices of the ordered basis of loops, Γ_0 (left) and Γ_1 (right)

$$R_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{if } \theta_j = 0, \quad j = 0, x, 1.$$

c) three monodromy matrices M_0, M_x, M_1 relative to the loops, similar to the matrices $\exp\{i\pi\theta_i\sigma_3\}\exp\{2\pi i R_i\}$, $i = 0, x, 1$, satisfying:

$$M_1\ M_x\ M_0 = e^{-i\pi\theta_\infty\sigma_3}e^{2\pi i R_\infty}$$

The data $\pm\theta_0, \pm\theta_x, \pm\theta_1, \pm(\theta_\infty - 1)$ are fixed by the equation. The other monodromy data are free. To each choice of them, there corresponds at least one fuchsian system (the solution of a Riemann-Hilbert problem for the given monodromy data). To such a fuchsian system, a branch $y(x)$ is associated. Therefore, there is a correspondence between a set of monodromy data and a branch $y(x)$. In generic cases, the correspondence is one-to-one. This is stated in the following theorem, proved in [17], section 3:

Let Θ, R, M stands for the collection $\theta_0, \theta_x, \theta_1, \theta_\infty \neq 0$, R_0, R_x, R_1, R_∞, M_0, M_x, M_1.

Theorem 1 *Let a basis of loops Γ be chosen and let the monodromy data with respect to Γ be Θ, R, M satisfying a), b), c) above. There is a one to one correspondence between the monodromy data and one branch of a transcendent $y(x)$, except when at least one $\theta_\nu \in \mathbf{Z}\backslash\{0\}$ and simultaneously $R_\nu = 0$. The branch in one to one correspondence with Γ, Θ, R, M will be denoted:*

$$y(x) = f_\Gamma(x; \Theta, R, M) \tag{9}$$

Note that for $\theta_j = 0$, M_j can be put in Jordan form $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$. Therefore:
There is a one to one correspondence if and only if one of the following conditions is satisfied:

(1) $\theta_\nu \notin \mathbf{Z}$, for every $\nu = 0, x, 1, \infty$;

(2) some $\theta_\nu \in \mathbf{Z}$ and $R_\nu \neq 0$, $\theta_\nu \neq 0$

(3) some $\theta_j = 0$ $(j = 0, x, 1)$ and simultaneously $\theta_\infty \notin \mathbf{Z}$, or $\theta_\infty \in \mathbf{Z}$ and $R_\infty \neq 0$.

Equivalently: *There is one to one correspondence except when one of the matrices M_i $(i = 0, x, 1)$, or $M_\infty^{-1} = M_1 M_x M_0$, is equal to $\pm I$.*

Define the following quantities:

$$p_\mu = \mathrm{Tr} M_\mu = 2\cos(\pi\theta_\mu), \quad p_{ij} = \mathrm{Tr}(M_i M_j), \quad \mu = 0, x, 1, \infty, \quad i, j \in \{0, x, 1\} \tag{10}$$

These coordinates describe the space of monodromy data, which is an affine cubic surface [19] [21]:

$$p_{0x}^2 + p_{01}^2 + p_{x1}^2 + p_{0x}p_{01}p_{x1} - (p_0 p_x + p_1 p_\infty)p_{0x} - (p_0 p_1 + p_x p_\infty)p_{01} - (p_x p_1 + p_0 p_\infty)p_{x1} +$$

$$+ p_0^2 + p_1^2 + p_x^2 + p_\infty^2 + p_0 p_x p_1 p_\infty - 4 = 0$$

The above relation follows by taking the trace of the relation $M_1 M_x M_0 M_\infty = I$.

If the monodromy group $< M_0, M_x, M_1 >$ is not reducible, or one of the matrices $M_0, M_x, M_1, M_1M_xM_0$ is not the identity, the above p_μ's, p_{ij}'s are a good system of coordinates for the monodromy group [19],[21].

As a consequence, a branch of a transcendent is uniquely parametrized by the p_μ's and p_{ij}'s to which it is in one to one correspondence. In other words, the integration constants are functions of p_μ's (or θ_μ's) and p_{ij}'s. The following notation expresses this dependence[4]:

$$y(x) = f_\Gamma(x; \theta_0, \theta_x, \theta_1, \theta_\infty, p_{0x}, p_{01}, p_{x1}) \tag{11}$$

A remarkable fact, established in Jimbo's paper [21], is that *this parametrization is explicit*, namely the integration constants are elementary or classical transcendental functions of the monodromy data.

As a consequence of this explicit parametrization of the three couples of integration constants at the three critical points in terms of *the same* monodromy data, the connection problem is solved. This is precisely the power of the method of monodromy preserving deformations.

We observe that, when the monodromy group is reducible, but none of the monodromy matrices $M_0, M_x, M_1, M_1M_xM_0$ is the identity, the one to one correspondence still holds, but the p_μ's, p_{ij}'s are no longer a good parametrization. The solutions in this case are known (see the Riccati solutions [33], [25]).

2.0.1 Analytic continuation of a branch

It is to be stressed that (9) (or (11)) represents *a branch* of a transcendent, for $|\arg x| < \pi$, $|\arg(1-x)| < \pi$, uniquely identified by the parametrization in terms of monodromy data Θ, R, M, which are associated to a the basis Γ. We show below that if the same monodromy data Θ, R, M are associated to another basis basis Γ', like Γ_0 and Γ_1 in figure 2, the parametrization $f_{\Gamma'}(x; \Theta, R, M)$ is the branch $y'(x)$ of the analytic continuation of $y(x)$ along a path in the x plane. Such path is the path that induces in the λ-plane (as x moves in the λ-plane around $\lambda = 0$ or 1) the deformation of the basis Γ into Γ'.

The two basis Γ_0 on the left and Γ_1 on the right of figure 2 can be regarded as the deformation of Γ, when x goes around a counterclockwise loop around $\lambda = 0$ or 1 respectively, namely when x goes counterclockwise along a path around $x = 0$ or $x = 1$ in the x-plane. The branch (9) undergoes its analytic continuation along these paths. *Being the deformation isomonodromic*, the monodromy matrices after the deformation do not change. So, the same M_0, M_x, M_1 are also assigned to $\gamma_0, \gamma_x, \gamma_1$ of the basis Γ_0 or Γ_1. Let $\tilde{y}(\tilde{x})$ represent the analytic continuation of $y(x)$ of (9). It is defined on the universal covering of points $\tilde{x}$ and can be written as $\tilde{y}(\tilde{x}) = f_\Gamma(\tilde{x}; \Theta, R, M)$. Its branch $y'(x)$ for $|\arg x| < \pi$, $|\arg(1-x)| < \pi$, has again a parametrization in terms of of M_0, M_x, M_1. But it differs form (9), because it is computed w.r.t. the basis Γ_0 or Γ_1. Let it be denoted by:

$$y'(x) = f_{\Gamma'}(x, \Theta, R, M) \qquad \text{where } \Gamma' \text{ stands for } \Gamma_0 \text{ or } \Gamma_1.$$

[4]The integration constants are two complex numbers. $\theta_0, \theta_x, \theta_1, \theta_\infty$ are fixed by the equation and p_{0x}, p_{01}, p_{x1} are not independent, becaus of the cubic surface relation. Accordingly, only two complex parameters are free.

One has now to compute $f_{\Gamma'}(x, \Theta, R, M)$. The way to do this is to compute the monodromy matrices associated to the basis Γ, being M_0, M_x, M_1 associated to Γ_0 or Γ_1. In order to do this, observe that the loops of figure 1 can be written as a product of the loops of figure 2 as:

$$\gamma_0(\text{ of figure 1}) = \gamma_x^{-1}\gamma_0\gamma_x(\text{ of figure 2, left}),$$

$$\gamma_x(\text{ of figure 1}) = \gamma_x^{-1}\gamma_0^{-1}\gamma_x\gamma_0\gamma_x(\text{ of figure 2, left}),$$

$$\gamma_1(\text{ of figure 1}) = \gamma_1(\text{ of left figure 2, left}),$$

and:

$$\gamma_0(\text{ of figure 1}) = \gamma_0(\text{ of figure 2, right}),$$

$$\gamma_x(\text{ of figure 1}) = \gamma_1^{-1}\gamma_x\gamma_1(\text{ of figure 2, right}),$$

$$\gamma_1(\text{ of figure 1}) = \gamma_1^{-1}\gamma_x^{-1}\gamma_1\gamma_x\gamma_1(\text{ of left figure 2, right})$$

It follows that, beeing $M_0, M_x M_1$ the monodromy matrices for the basis Γ_0 or Γ_1, the monodromy matrices with respect to the initial basis Γ are:

$$M_0' = M_x M_0 M_x^{-1}, \quad M_x' = M_x M_0 M_x M_0^{-1} M_x^{-1}, \quad M_1' = M_1, \tag{12}$$

in the left case (basis Γ_0, counterclockwise loop of x around 0).

$$M_0' = M_0, \quad M_x' = M_1 M_x M_1^{-1}, \quad M_1' = M_1 M_x M_1 M_x^{-1} M_1^{-1}, \tag{13}$$

in the right case (basis Γ_1, counterclockwise loop of x around 1). The above transformation of the monodromy matrices is an action of the *braid group*. It implies that Θ and R are not changed. The branch of the analytic continuation is then:

$$y'(x) = f_\Gamma(x; \Theta, R, M')$$

and thus the computation of $f_{\Gamma'}(x; \Theta, R, M)$ has been completed. To summarize:

Let Θ, R, M be given. The choice of the basis Γ determines a branch $y(x) = f_\Gamma(x; \Theta, R, M)$. The choice of another basis Γ' determines another branch $y'(x) = f_{\Gamma'}(x; \Theta, R, M)$, which is a branch of the analytic continuation of $y(x)$ along the path of x which deforms Γ to Γ'. The relation is

$$f_{\Gamma'}(x; \Theta, R, M) = f_\Gamma(x; \Theta, R, M')$$

where $M \mapsto M'$ is an action of the braid group generated by (12) and (13). In other words, the anayltic continuation of $y(x) = f_\Gamma(x; \Theta, R, M)$ is $y'(x) = f_\Gamma(x; \Theta, R, M')$.

In terms of the coordinates p_μ's and p_{ij}"s the above transformation of the matrices reads:

$$\begin{cases} p_{x1}' = p_{x1}(p_{0x}^2 - 1) + p_{0x}p_{01} - (p_\infty p_x + p_1 p_0)p_{0x} + p_\infty p_0 + p_1 p_x \\ \\ p_{0x}' = p_{0x}, \quad p_{01}' = -p_{01} - p_{x1}p_{0x} + p_\infty p_x + p_1 p_0 \end{cases} \tag{14}$$

in the left case (basis Γ_0, x goes around a loop around 0), and:

$$\begin{cases} p'_{01} = p_{01}(p_{x1}^2 - 1) + p_{0x}p_{x1} - (p_\infty p_1 + p_0 p_x)p_{x1} + p_\infty p_x + p_0 p_1 \\ \\ \qquad p'_{1x} = p_{1x}, \quad p'_{0x} = -p_{0x} - p_{01}p_{x1} + p_\infty p_1 + p_0 p_x \end{cases} \tag{15}$$

in the right case (basis Γ_1, x goes around a loop around 1). The branch of the analytic continuation has parametrization:

$$y'(x) = f_\Gamma(x; \theta_0, \theta_x, \theta_1, \theta_\infty, p'_{0x}, p'_{01}, p'_{x1}), \qquad |\arg x| < \pi, \quad |\arg(1-x)| < \pi.$$

The parametrization of the branch of the analytic continuation along more complicated paths is given by a suitable composition of (14) and (15).

As a final remark it is to be noted that the choice $|\arg x + 2\pi k| < \pi$, $|\arg(1-x) + 2\pi l| < \pi$, for some $k, l \in \mathbf{Z}$, is also possible. But the computation of the *explicit* parametrization is done by the procedure of [21], which makes use of a reduction of (1) to hyper-geometric equations (and by its generalization of [16] to non-hyper-geometric reductions in case of Taylor solutions). This computation requires that $k = l = 0$. Accordingly, the formulas which parametrize the critical behaviors in this paper are given for the branches $|\arg x| < \pi$, $|\arg(1-x)| < \pi$.

3 Critical behavior at $x = 0$

In the following, it is understood that $x \to$ critical point inside a sector. The behavior of $y(x)$ at $x = 0, 1, \infty$ is determined by three *critical exponents* $\sigma_0, \sigma_1, \sigma_\infty$ respectively, given by:

$$2\cos(\pi\sigma_0) = p_{0x}, \quad 2\cos(\pi\sigma_1) = p_{x1}, \quad 2\cos(\pi\sigma_\infty) = p_{01}, \quad 0 \le \Re\sigma_i \le 1,$$

where p_{ij} are (10).

Remark: The above relation determines σ_i up to $\sigma_i \mapsto \pm\sigma_i + 2n$, $n \in \mathbf{Z}$. One can then restrict to the case $0 \le \Re\sigma_i \le 1$, as it is explained in [14] [15]. Despite this condition, when $\Re\sigma_i = 0$ the ambiguity of sign cannot be eliminated. Namely:

$$\sigma_i = \pm i\nu, \quad \nu \in \mathbf{R}, \qquad p_{kl} = \cosh \pi\nu > 2,$$

In case $\Re\sigma_i = 1$ the ambiguity $\sigma_i \mapsto 2 - \sigma_i$ cannot be eliminated. Namely:

$$\sigma_i = 1 \pm i\nu, \quad \nu \in \mathbf{R}, \qquad p_{kl} = -\cosh \pi\nu < -2,$$

Anyway, a solution $y(x)$ corresponding to such monodromy data is invariant for the change of sign of ν, as it will be explained below.

We start with the critical point $x = 0$. In the following, we use the notation $\sigma := \sigma_0$. Let also $|x| < \epsilon < 1$, where ϵ is sufficiently small for all our purposes. The first result of this paper is the following:

Proposition 1 [**Inverse sine-type oscillatory behaviors**] *The equation PVI admits solutions with a branch at $x=0$ behaving in the following way when $x\to 0$, with arg(x) bounded:*

$$y(x)=\frac{1}{iA\sin\Bigl(i(1-\sigma)\ln x+\phi\Bigr)+B+\delta^*(x)},\qquad \delta^*(x)=O(x) \tag{16}$$

where $\sigma,\phi\in\mathbf{C}$ are the integration constants, satisfying $\Re\sigma=1$, $\sigma\neq 1$. The coefficients A and B are:

$$B=\frac{\nu^2+2\gamma-2\alpha}{2\nu^2}=\frac{\nu^2+(\theta_1)^2-(\theta_\infty-1)^2}{2\nu^2},\qquad \sigma=1+i\nu,\quad \nu\in\mathbf{R},\quad \nu\neq 0,$$

$$A=i\sqrt{\frac{2\alpha}{\nu^2}+B^2}=i\sqrt{\frac{(\theta_\infty-1)^2}{\nu^2}+B^2}= \tag{17}$$

$$=\frac{\sqrt{\Bigl[(1-\sigma)^2-(\theta_\infty-1-\theta_1)^2\Bigr]\Bigl[(\theta_\infty-1+\theta_1)^2-(1-\sigma)^2\Bigr]}}{2(1-\sigma)^2}$$

The vanishing term $\delta^*(x)$ has convergent expansion for $0<|x|<\epsilon$:

$$\delta^*(x)=\sum_{n=1}^{\infty}x^n\sum_{m=-n-1}^{n+1}d_{nm}\;x^{m(1-\sigma)}\;=\sum_{m_1=1}^{\infty}\sum_{m_2=-1}^{2m_1+1}e_{m_1m_2}x^{m_1\sigma}x^{m_2(1-\sigma)} \tag{18}$$

$$e_{m_1m_2}=d_{m_1,m_2-m_1}.$$

The coefficients are certain rational functions of σ and $\exp\{i\phi\}$, which can be computed by direct substitution into PVI (see section 7). The constant σ is related to the monodromy data associated to $y(x)$ by: $2\cos(\pi\sigma)=p_{0x}<-2$.

Since $\sin(2x)=1-2\sin^2(x-\pi/4)$, we can also rewrite:

$$y(x)=\left\{-2iA\sin^2\left(i\frac{1-\sigma}{2}\ln x+\frac{\phi}{2}-\frac{\pi}{4}\right)\;+iA+B\;+\delta^*(x)\right\}^{-1} \tag{19}$$

Let $r\in\mathbf{C}$, $r\neq 0$. It is convenient, for future developments, to re-parametrize ϕ in terms of r as follows (at this stage of the discussion, this may be temporarily taken as the definition of r):

$$\phi=i\ln\frac{2r}{(1-\sigma)A}$$

The reason to introduce r is that it is a natural parameter that will be written in section 5 as a function of the monodromy data associated to the basis Γ of figure 1:

$$r=r(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty,p_{x1},p_{01}).$$

This parametrization identifies uniquely the branch.

The sign of the square root A can be chosen arbitrarily, because it changes $\phi\mapsto\phi+(2k+1)\pi$, and $y(x)$ is invariant. It is to be noted that the condition $\Re\sigma=1$ does not fix the ambiguity $\sigma=1+i\nu\mapsto 1-i\nu$, (namely $\sigma\mapsto 2-\sigma$), but the substitution

$\sigma \mapsto 2-\sigma$ induces $\phi \mapsto -\phi+(2k+1)\pi$ (see Appendix I, subsection 8.1.1), and thus $y(x)$ is invariant.

Remark: *We must keep $\delta(x)$ in the denominator.* This term is essential in that it determines the position of the movable poles, which occur when the denominator vanishes at some isolated points.

For completeness, the results about the critical behavior when $0 \leq \Re\sigma < 1$, $\sigma \neq 0$, are reported below. Though the critical behaviors are already known and appear in [21], [9] [14] [15] [16], the expansions of the terms $\delta(x)$ and $\delta^*(x)$ in the propositions below is a result of the present paper (see section 7).

Proposition 2 [**Small power type behaviors (Jimbo)**] *The equation PVI admits solutions with a branch having the following behavior, when $x \to 0$, arg(x) bounded ([21], [9] [14] [15] [16]):*

$$y(x) = a x^{1-\sigma}\ (1+\delta(x)), \quad \delta(x) = O(\max\{x^{1-\Re\sigma}, x^{\Re\sigma}\}) \tag{20}$$

where $a, \sigma \in \mathbf{C}$ are integration constants such that $a \neq 0$ and $0 < \Re\sigma < 1$. The higher order term $\delta(x)$ has the following convergent expansion for $0<|x|<\epsilon$ (section 7):

$$\delta(x) = -1 + \sum_{n=0}^{\infty} x^n \sum_{m=-n}^{n+2} \tilde{c}_{nm} x^{m\sigma}, \quad \tilde{c}_{00}=1$$

We can also write:

$$\delta(x) = \sum_{m_2=0}^{\infty} \sum_{m_1=0}^{2m_2+2} \delta_{m_1 m_2} x^{m_1\sigma} x^{m_2(1-\sigma)}, \qquad m_1+m_2 \geq 1 \tag{21}$$

$$\delta_{m_1 m_2} = \tilde{c}_{m_2, m_1-m_2}.$$

The coefficients are certain rational functions of σ and a, which can be computed by direct substitution into PVI (see section 7). The exponent σ is related to the monodromy data associated to $y(x)$ by: $2\cos(\pi\sigma) = p_{0x}$.

As before, we re-parameterize a in terms of a new $r \in \mathbf{C}$:

$$a = \frac{1}{16\sigma^3 r}\Big[\sigma^2 - (\sqrt{-2\beta} - \sqrt{1-2\delta}\ )^2\Big]\Big[(\sqrt{-2\beta}+\sqrt{1-2\delta}\ )^2 - \sigma^2\Big] =$$

$$= \frac{[\sigma^2-(\theta_0-\theta_x)^2][(\theta_0+\theta_x)^2-\sigma^2]}{16\sigma^3 r}. \tag{22}$$

r will be naturally introduced when proving (20) in Appendix I. The parametrization of r in terms of monodromy data identifies the branch uniquely.

Remark: For special values of σ we have the following solutions:

$$y(x) = \frac{\theta_0}{\theta_0+\theta_x}\, x \ \mp\ \frac{r}{\theta_0+\theta_x}\, x^{1+\sigma} + O(x^2), \quad \sigma = \pm(\theta_0+\theta_x) \neq 0, \tag{23}$$

$$y(x) = \frac{\theta_0}{\theta_0-\theta_x}\, x \ \mp\ \frac{r}{\theta_0-\theta_x}\, x^{1+\sigma} + O(x^2), \quad \sigma = \pm(\theta_0-\theta_x) \neq 0. \tag{24}$$

Proposition 3 [**Sine-type oscillatory behaviors**] *The equation PVI admits solutions with a branch having the following behavior, when $x \to 0$, arg(x) bounded ([21], [16]):*

$$y(x) = x\ \Big\{ iA \sin(i\sigma \ln x + \phi) + B + \delta^*(x) \Big\}, \quad \delta^*(x) = O(x) \tag{25}$$

where $\sigma, \phi \in \mathbf{C}$ are integration constants such that $\Re\sigma = 0$, $\sigma \neq 0$. The coefficients are:

$$B = \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} = \frac{\sigma^2 - 2\beta - 1 + 2\delta}{2\sigma^2},$$

$$A = \frac{\sqrt{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}}{2\sigma^2} = \sqrt{\frac{\theta_0^2}{\sigma^2} - B^2} = \sqrt{-\frac{2\beta}{\sigma^2} - B^2},$$

The term $\delta^*(x)$ has convergent expansion for $0 < |x| < \epsilon$:

$$\delta^*(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n-1}^{n+1} b_{nm} x^{m\sigma} = \sum_{m_2=1}^{\infty} \sum_{m_1=-1}^{2m_2+1} a_{m_1 m_2} x^{m_1\sigma} x^{m_2(1-\sigma)}, \tag{26}$$

$$a_{m_1 m_2} = b_{m_2, m_1 - m_2}.$$

The constant σ is related to the monodromy data associated to $y(x)$ by: $2\cos(\pi\sigma) = p_{0x} > 2$.

The critical behavior can be also written as:

$$y(x) = x \left\{ -2iA \sin^2\left(i\frac{\sigma}{2} \ln x + \frac{\phi}{2} - \frac{\pi}{4} \right) \ + iA + B + \delta^*(x) \right\} \tag{27}$$

We rewrite ϕ in terms of the new integration constant r, which will be expressed as $r = r(\sigma, \theta_0, \theta_x, \theta_1, \theta_\infty, p_{x1}, p_{01})$ in section 5 :

$$\phi = i \ln \frac{2r}{\sigma A}$$

Any sign of the square root in A can be chosen (change $\phi \mapsto \phi + (2k+1)\pi$). The condition $\Re\sigma = 0$ does not fix the ambiguity $\sigma \mapsto -\sigma$. Nevertheless, (25) is invariant for $\sigma \mapsto -\sigma$, because this induces the change $\phi \mapsto -\phi + (2k+1)\pi$, $k \in \mathbf{Z}$. See the Appendix I (at the end of subsection 8.1.1) for the proof.

Remark: If $\Re\sigma = 0$ ($\sigma = i\nu$, $\nu \in \mathbf{R}$) but $\sigma \in \{\theta_0 + \theta_x, \theta_0 - \theta_x, -\theta_0 + \theta_x, -\theta_0 - \theta_x\}$, the solution of proposition 3 becomes:

$$y(x) = x \left(\frac{\theta_0}{i\nu} - \frac{r}{i\nu} x^{i\nu} \right) + O(x^2), \qquad \sigma = \theta_0 \pm \theta_x, \tag{28}$$

$$y(x) = x \left(-\frac{\theta_0}{i\nu} - \frac{r}{i\nu} x^{i\nu} \right) + O(x^2), \qquad \sigma = -(\theta_0 \pm \theta_x) \tag{29}$$

How we Prove the above propositions:

(20) and (25) are proved (though not explicitly written) in [21]. The proof is reviewed in Appendix I.

Formula (16) is proved in section 4, where we show that it is the image of (27) via a fractional linear transformation (30).

In section 7 the recursive procedure is given to compute the full expansion of $y(x)$, and thus the series (18), (21), (26). Their convergence follows from the elliptic representation. The elliptic representation of PVI is analytically studied in [15]. All the critical behaviors of $y(x)$ are computed for $0 \leq \Re\sigma \leq 1$, $\sigma \neq 0, 1$. The convergence of the full expansions is proved.

– When $0 < \Re\sigma < 1$, the critical behavior and the full convergent expansion obtained from the elliptic representation ((60) in Appendix II), coincides with (20). This proves the convergence of (21).

– When $\Re\sigma = 0$ and $\Re\sigma = 1$, the critical behaviors computed in [15] depend on two integration constants σ, ϕ_E (three real constants). They are (see Appendix II):

$$y(x) = x\left[\sin^2\left(i\frac{\sigma}{2}\ln x + \phi_E + \sum_{n\geq 1} c_n(\sigma)[e^{-2i\phi_E}x^{\sigma}]^n\right) + \delta^*_E(x)\right],$$

$$\Re\sigma = 0, \quad |x| < \epsilon, \quad |e^{-2i\phi_E}x^{\sigma}| < \epsilon, \quad \delta^*_E(x) = O(x).$$

$$y(x) = \left[\sin^2\left(i\frac{1-\sigma}{2}\ln x + \phi_E + \sum_{n\geq 1} c_n(\sigma)[e^{-2i\phi_E}x^{1-\sigma}]^n\right) + \delta^*_E(x)\right]^{-1},$$

$$\Re\sigma = 1, \quad |x| < \epsilon, \quad |e^{-2i\phi_E}x^{1-\sigma}| < \epsilon, \quad \delta^*_E(x) = O(x).$$

The series in $\sin^2(\ ..\ )$ are absolutely convergent for sufficiently small $r < 1$. They are oscillating series that do not vanish when $x \to 0$. In Appendix II the convergent expansion of the terms $\delta^*_E(x)$ is also given. In subsection 9.3 of Appendix II, the reader finds the proofs that the above behaviors coincide with our (27) and (19). In order to do this, first write $\sigma = -i\nu$ or $1 + i\nu$, $\nu \in \mathbf{R}$. Then, it is shown that:

$$-2iA\sin^2\left(\frac{\nu}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) + iA + B = \sin^2\left(\frac{\nu}{2}\ln x + f(x)\right)$$

where $f(x)$ is an oscillating function:

$$f(x) = \sum_{n\geq 0} f_n x^{-i\nu x}, \quad f_n \in \mathbf{C}.$$

The coincidence of the result of the present paper with that of the elliptic representation, together with the convergence of the expansions of $\delta^*_E(x)$, proves the convergence of (18) and (26).

4 Proof of Proposition 1. The critical behavior at $x = 0$ when $\Re\sigma = 1$. A Fractional Linear Transformation

We consider the following fractional linear transformation, studied in [10]:

$$\theta'_0 = \theta_\infty - 1, \quad \theta'_x = \theta_1, \quad \theta'_1 = \theta_x, \quad \theta'_\infty = \theta_0 + 1; \qquad y'(x') = \frac{x}{y(x)}, \quad x' = x. \tag{30}$$

$$(p_0, p_x, p_1, p_\infty; p_{0x}, p_{01}, p_{x1}) \mapsto$$
$$\mapsto (p'_0, p'_x, p'_1, p'_\infty; p'_{0x}, p'_{01}, p'_{x1}) = (-p_\infty, p_1, p_x, -p_0; -p_{0x}, -p_{01}, p_{x1}).$$

This is a symmetry of PVI, namely $y(x)$ solves PVI with coefficients θ_μ if and only if y' solves PVI with coefficients θ'_μ. We are going to use this transformation to obtain the critical behavior of a transcendent $y'(x)$ with $\Re\sigma' = 1$, $p'_{0x} < -2$, from the behavior of a transcendent $y(x)$ with $p_{0x} = -p'_{0x} > 2$, $\Re\sigma = 0$.

⋄ First, we compute the relation between σ' and σ. The relation $p'_{0x} = -p_{0x}$ implies:

$$2\cos(\pi\sigma') = -2\cos(\pi\sigma) \implies \sigma' = \pm\sigma + (2k+1)\pi, \quad k \in \mathbf{Z}$$

The conditions $0 \leq \Re\sigma < 1$, $0 \leq \Re\sigma' < 1$ imply that:

$$\sigma' = 1 - \sigma \tag{31}$$

⋄ We compute the solution $y'(x)$ with $\Re\sigma' = 1$ from the solution $y(x)$ with $\Re\sigma = 0$. We know the critical behavior of this solution from the Jimbo's procedure of Appendix I:

$$y(x) = x\ \{iA\sin(i\sigma\ln x + \phi) + B + \delta^*(x)\}, \quad \sigma = \pm i\nu,$$

$$\phi = i\ln\frac{2r}{\sigma A}, \qquad B = \frac{\nu^2 + \theta_x^2 - \theta_0^2}{2\nu^2} \equiv \frac{\nu^2 + (\theta'_1)^2 - (\theta'_\infty - 1)^2}{2\nu^2}$$

$$A = i\sqrt{\frac{\theta_0^2}{\nu^2} + \left[\frac{\nu^2 + \theta_x^2 - \theta_0^2}{2\nu^2}\right]^2} \equiv i\sqrt{\frac{(\theta'_\infty - 1)^2}{\nu^2} + \left[\frac{\nu^2 + (\theta'_1)^2 - (\theta'_\infty - 1)^2}{2\nu^2}\right]^2}$$

The solution $y' = \frac{x}{y}$ obtained by fractional linear transf. from $y(x)$ is immediately computed:

$$y'(x) = \Big\{iA\sin\Big(i(1-\sigma')\ln x + \phi\Big) + B + \delta^*(x)\Big\}^{-1}, \quad \sigma' = 1 \mp i\nu,$$

$$\phi = i\ln\frac{2r}{(1-\sigma')A}, \quad \delta^*(x) = O(x)$$

In section 7 we compute the full expansion of $y(x)$, which proves that $\delta^*(x)$ has the form (26). As a result, the expansion of $y'(x)$ obtained by the fractional linear transformation proves (18) from (26).

4.1 The case $\Re\sigma = 1$ associated to a Frobenius Manifold

PVI is associated to a Frobenius Manifold when $\theta_0 = \theta_x = \theta_1 = 0$ [8]. The result of the general case, when $\Re\sigma = 1$, becomes:

$$B = \frac{\nu^2 - (\theta_\infty - 1)^2}{2\nu^2}, \qquad A = \pm i \frac{\nu^2 + (\theta_\infty - 1)^2}{2\nu^2}, \qquad \sigma = 1 + i\nu, \quad \nu \in \mathbf{R}$$

If we choose the minus sign in A, then $iA + B = 1$, and:

$$y(x) = \left\{ 1 - \frac{\nu^2 + (\theta_\infty - 1)^2}{\nu^2} \sin^2\left(i\frac{1-\sigma}{2} \ln x + \frac{\phi}{2} - \frac{\pi}{4} \right) + \delta^*(x) \right\}^{-1}$$

$$\phi = i \ln \frac{4r\nu^2}{i(\sigma - 1)(\nu^2 + (\theta_\infty - 1)^2)}.$$

If we choose the plus sign in A, then $iA + B = -(\theta'_\infty - 1)^2/\nu^2$, and:

$$y(x) = \left\{ \frac{\nu^2 + (\theta_\infty - 1)^2}{\nu^2} \sin^2\left(i\frac{1-\sigma}{2} \ln x + \frac{\varphi}{2} - \frac{\pi}{4} \right) - \frac{(\theta_\infty - 1)^2}{\nu^2} + \delta^*(x) \right\}^{-1}$$

$$\varphi = \phi + (2k+1)\pi, \quad k \in \mathbf{Z}$$

The two ways of writing $y'(x)$ give the same solution (verify using $\sin^2 = 1 - \cos^2$).

5 Behaviors at $x = 1, \infty$. Connection problem

In this section is computed the behavior at $x = 1$ and $x = \infty$ of a solution with $p_{x1} < -2$ (i.e. $\Re\sigma_1 = 1$) and $p_{01} < -2$ (i.e. $\Re\sigma_\infty = 1$) respectively. Also the formulae which allow to solve the connection problem are computed. The results are in Proposition 6 and Proposition 7.

In order to understand the results, it is necessary to review the general scheme and formulae to solve the connection problem for $0 \leq \Re\sigma_i < 1$. In doing this, for the first time in the literature the general formulas are given (namely, the coefficients $\mathbf{G}_i$ in (33)) which express the monodromy data associated to a solution, in terms of the coefficients of PVI and of the integration constants of the solution.

5.1 Formulae of the Relation between Monodromy Data and Integration Constants

The integration constants σ and r in (20) and (25) are functions of the monodromy data. These functions are computed in [21]. Due to a miss print in [21], the correct expression is re-computed in [1], and the result is as follows:

$$2\cos\pi\sigma = p_{0x}$$

$$
\begin{aligned}
r &= r(\theta_0,\theta_x,\theta_1,\theta_\infty;\sigma,p_{01},p_{x1})\\
&= \frac{(\theta_0-\theta_x+\sigma)(\theta_0+\theta_x-\sigma)(\theta_\infty+\theta_1-\sigma)}{4\sigma(\theta_\infty+\theta_1+\sigma)}\ \frac{1}{\mathbf{F}},
\end{aligned}
\tag{32}
$$

where

$$
\mathbf{F} := \frac{\Gamma(1+\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma)+1\right)}{\Gamma(1-\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma)+1\right)}\times
$$

$$
\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma)+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma)+1\right)}\ \frac{V}{U},
$$

and:

$$
U := \left[\frac{i}{2}\sin(\pi\sigma)p_{x1}-\cos(\pi\theta_x)\cos(\pi\theta_\infty)-\cos(\pi\theta_0)\cos(\pi\theta_1)\right]e^{i\pi\sigma}\ +
$$

$$
+\frac{i}{2}\sin(\pi\sigma)p_{01}+\cos(\pi\theta_x)\cos(\pi\theta_1)+\cos(\pi\theta_\infty)\cos(\pi\theta_0)
$$

$$
V := 4\sin\frac{\pi}{2}(\theta_0+\theta_x-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_x+\sigma)\ \sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma).
$$

Remarks:

1) Formula (32) was computed with the assumption that $\theta_0,\theta_x,\theta_1,\theta_\infty,\sigma$ are not integers, and $\sigma\pm(\theta_0+\theta_x)$, $\sigma\pm(\theta_0-\theta_x)$, $\sigma\pm(\theta_1+\theta_\infty)$, $\sigma\pm(\theta_1-\theta_\infty)$ are not even integers. Formula (32) has finite non vanishing limit when σ tends to $\pm(\theta_0+\theta_x)$, $\pm(\theta_0-\theta_x)$. The corresponding solutions are (23) and (24).

2) In the case $\theta_0=\theta_x=\theta_1=0$, r is computed in [9] for the generic case, and in [14] *for all possible values of* $\theta_\infty\neq 0$ and $\sigma\not\in(-\infty,0)\cup[1,\infty)$.

As for logarithmic solutions and Taylor expanded solutions, the parametrization of the integration constant r in temrs of monodromy data is given in [16] [17] (and also in [21] for the τ function of the logarithmic case). In this cases, $\sigma=0$ ($p_{ij}=2$) and $\sigma=1$. ($p_{ij}=-2$), Please, refer to these papers for the results. Here, we concentrate on Jimbo's solutions and the sine-type oscillatory behaviors, for which $\sigma\neq 0,1$.

In order to solve the connection problem, also the inverse formulae of (32) are necessary, which give p_{0x},p_{01},p_{x1} in terms of σ, r and the coefficients of PVI, namely $\theta_0,\theta_x,\theta_1,\theta_\infty$. To compute p_{0x},p_{01},p_{x1}, one has to starts from the monodromy matrices, which have been computed in [21] for the first time, and subsequently in [15], [1]. Taking their traces, we obtain:

$$
\begin{cases}
p_{0x} &= \quad 2\cos\pi\sigma\\[2ex]
p_{x1} &= \quad \mathbf{G}_1 r^{-1}+\mathbf{G}_2+\mathbf{G}_3 r\\[2ex]
p_{01} &= \quad \mathbf{G}_4 r^{-1}+\mathbf{G}_5+\mathbf{G}_6 r
\end{cases}
\tag{33}
$$

where $\mathbf{G}_i$ are rational functions of $\theta_\mu\pm\theta_\nu\pm\sigma$, $\cos\pi\sigma$, $\cos\pi\theta_\mu$, $\Gamma((\theta_\mu\pm\theta_\nu\pm\sigma)/2)$, $e^{\pm i\pi\sigma}$. The explicit computation of the coefficients $\mathbf{G}_i$'s is very complicated, and it is not written

anywhere (except for [9] and [14], when $\theta_0=\theta_x=\theta_1=0$). So, here the G_i's are given, for the first time. To do the computation, the monodromy matrices of [15], page 1355-1357, formulae (A23), (A24), (A25) (r appears with the name s) are used, with the assumption that $\theta_0,\theta_x,\theta_1,\theta_\infty,\sigma$ are not integers. Here is the result.

Let $s(z):=\sin(\frac{\pi}{2}z)$ and

$$\Xi=\Big(s(\theta_0+\theta_x+\sigma)s(\theta_0-\theta_x-\sigma)+s(\theta_0-\theta_x+\sigma)s(\theta_0+\theta_x-\sigma)\Big)\times$$
$$\times\Big(s(\theta_1+\theta_\infty+\sigma)s(\theta_1-\theta_\infty+\sigma)+s(\theta_1+\theta_\infty-\sigma)s(\theta_1-\theta_\infty-\sigma)\Big)$$

$$\Xi_1=\Big(s(\theta_0+\theta_x+\sigma)s(\theta_0-\theta_x+\sigma)+s(\theta_0+\theta_x-\sigma)s(\theta_0-\theta_x-\sigma)\Big)\times$$
$$\times\Big(s(\theta_1+\theta_\infty+\sigma)s(\theta_1-\theta_\infty+\sigma)+s(\theta_1+\theta_\infty-\sigma)s(\theta_1-\theta_\infty-\sigma)\Big)$$

$$\Omega=\Big(-s(\theta_0+\theta_x+\sigma)s(\theta_0-\theta_x-\sigma)+s(\theta_0-\theta_x+\sigma)s(\theta_0+\theta_x-\sigma)\Big)\times$$
$$\times\Big(s(\theta_1+\theta_\infty+\sigma)s(\theta_1-\theta_\infty+\sigma)-s(\theta_1+\theta_\infty-\sigma)s(\theta_1-\theta_\infty-\sigma)\Big)$$

$$\Omega_1=\Big(s(\theta_0+\theta_x+\sigma)s(\theta_0-\theta_x+\sigma)-s(\theta_0+\theta_x-\sigma)s(\theta_0-\theta_x-\sigma)\Big)\times$$
$$\times\Big(s(\theta_1+\theta_\infty+\sigma)s(\theta_1-\theta_\infty+\sigma)-s(\theta_1+\theta_\infty-\sigma)s(\theta_1-\theta_\infty-\sigma)\Big)$$

Let also:

$$\mathcal{F}:=\frac{\Gamma(1+\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma)+1\right)}{\Gamma(1-\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma)+1\right)}\times$$

$$\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma)+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma)+1\right)}\,\frac{4\sigma(\theta_\infty+\theta_1+\sigma)}{(\theta_0-\theta_x+\sigma)(\theta_0+\theta_x-\sigma)(\theta_\infty+\theta_1-\sigma)}$$

and:

$$V:=4\sin\frac{\pi}{2}(\theta_0+\theta_x-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_x+\sigma)\ \sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma).$$

$$V_1:=V(\sigma\mapsto-\sigma)$$

The result is:

$$\mathbf{G}_2=\frac{2(\Omega\ \cos\pi\theta_x\cos\pi\theta_1\ -\ \Xi\ \sin\pi\theta_x\sin\pi\theta_1)}{\sin^2(\pi\sigma)\sin\pi\theta_x\sin\pi\theta_1};$$

$$\mathbf{G}_5=2\Big(\cos\pi\theta_1\cos\pi\theta_0+\frac{\Xi_1}{\Omega_1}\ \sin\pi\theta_1\sin\pi\theta_0\Big);$$

and

$$\mathbf{G}_1=-\frac{\sin\pi\theta_x\sin\pi\theta_1}{\Omega}\ V_1\ \frac{1}{\mathcal{F}},\qquad \mathbf{G}_3=-\frac{\sin\pi\theta_x\sin\pi\theta_1}{\Omega}\ V\ \mathcal{F}$$

$$\mathbf{G}_4 = -e^{i\pi\sigma}\frac{\sin\pi\theta_0}{\sin\pi\theta_x}\frac{\Omega}{\Omega_1}\mathbf{G}_1, \qquad \mathbf{G}_6 = -e^{-i\pi\sigma}\frac{\sin\pi\theta_0}{\sin\pi\theta_x}\frac{\Omega}{\Omega_1}\mathbf{G}_3.$$

Observe that the limit of the $\mathbf{G}_i$'s exists also for $\sigma \pm (\theta_0 \pm \theta_x) \to 2n$, $n \in \mathbf{Z}$, though this is not always the case for the solution (32) (which has anyway limit for $\sigma \pm (\theta_0 \pm \theta_x) \to 0$.

5.2 Critical behaviors at $x = 1, \infty$ from the behavior at $x = 0$

One can avoid recomputing the critical behaviors at $x = 1, \infty$. They can be deduced from the behaviors at $x = 0$ with two fractional linear transformations, which are symmetries of PVI.

The transformation σ_{01} exchanges the values 0 and 1 of the independent variable:

$$\sigma_{01}: \quad \theta_0' = \theta_1, \;\; \theta_x' = \theta_x, \;\; \theta_1' = \theta_0, \;\; \theta_\infty' = \theta_\infty; \quad y'(x') = 1 - y(x), \quad x' = 1 - x. \tag{34}$$

Therefore, when $x \to 0$ then $x' \to 1$. We obtain the behavior of $y'(x')$ at $x' = 1$ from that of $y(x)$ at $x = 0$. The monodromy data change as follows [17]:

$$\begin{cases} p_{01}' &= -p_{01} - p_{0x}p_{x1} + p_\infty p_x + p_1 p_0 \\ p_{0x}' &= p_{x1} \\ p_{x1}' &= p_{0x} \end{cases}$$

and the inverse:

$$\begin{cases} p_{01} &= -p_{01}' - p_{0x}'p_{x1}' + p_\infty' p_x' + p_1' p_0' \\ p_{0x} &= p_{1x}' \\ p_{x1} &= p_{0x}' \end{cases} \tag{35}$$

This means that y' is associated to the monodromy data with $'$. Namely:

$$y'(x', \Theta', P') = 1 - y\Big(x(x'), \Theta(\Theta'), P(P')\Big)$$

where Θ stands for the collection of θ_μ's, and P for the collection of the p_{ij}'s. Formula (34) gives $x(x')$ and $\Theta = \Theta(\Theta')$, while $P(P')$ is (35).

The transformation σ_{x1} exchanges the values x and 1 of the independent variable:

$$\sigma_{x1}: \quad \theta_x' = \theta_1, \;\; \theta_1' = \theta_x; \qquad \theta_0' = \theta_0, \;\; \theta_\infty' = \theta_\infty; \qquad y'(x') = \frac{1}{x}y(x), \quad x' = \frac{1}{x}.$$

Therefore, if $x \to 0$, $x' \to \infty$ and we obtain the behavior of $y'(x')$ from that of $y(x)$. The monodromy data change [17] [26] as follows:

$$\begin{cases} p_{0x}' &= -p_{01} - p_{0x}p_{x1} + p_\infty p_x + p_0 p_1 \\ p_{01}' &= p_{0x} \\ p_{1x}' &= p_{1x} \end{cases}$$

Namely:

$$\begin{cases} p_{01} &=& -p'_{0x} - p'_{01}p'_{x1} + p'_{\infty}p'_1 + p'_0p'_x \\ \\ p_{0x} &=& p'_{01} \\ \\ p_{1x} &=& p'_{1x} \end{cases} \tag{36}$$

Remark: The proof of (35) and (36) see [17]. The result depends on the choice of the base of loops for the fuchsian system associated to $y'(x')$. Different choices of loops that preserve the ordering $1, 2, 3$ for $0, x', 1$ correspond to different branches of $y'(x')$. The choice of the basis of loops in [17], which gives (35) and (36), is actually the choice that gives the simplest form for p'_{ij}. All other possible values of p'_{ij} can be obtained from (35) and (36) by the action of the braid group generated by (14) and (15).

5.3 Parametrization Formulae when $0 < \Re\sigma_i < 1$, $p_{ij} \notin (-\infty, -2] \cup [2, \infty)$

Proposition 4 *Let PVI be give, namely $\theta_0, \theta_x, \theta_1, \theta_\infty$ are given. Let the basis Γ of figure 1 be chosen, so that the monodromy data are refered to Γ. If $p_{0x}, p_{01}, p_{x1} \notin (-\infty, -2] \cup [2, \infty)$, there exists a transcendent whith branches at $x = 0, 1, \infty$ having the following asymptotic behaviors:*

$$y(x) = \begin{cases} a_0 x^{1-\sigma_0}(1 + \delta_0(x)), \quad x \to 0 \\ \\ 1 - a_1(1-x)^{1-\sigma_1}(1 + \delta_1(1-x)), \quad x \to 1 \\ \\ a_\infty x^{\sigma_\infty}(1 + \delta_\infty(x^{-1})), \quad x \to \infty \end{cases} \tag{37}$$

The branches at $x = 0$, 1 and ∞ are identified by the following parametrization:

$$2\cos(\pi\sigma_0) = p_{0x}, \quad 2\cos(\pi\sigma_1) = p_{x1}, \quad 2\cos(\pi\sigma_\infty) = p_{01}. \tag{38}$$

$$a_0 = \frac{[\sigma_0^2 - (\theta_0 - \theta_x)^2][(\theta_0 + \theta_x)^2 - \sigma_0^2]}{16\sigma_0^3 r_0} \tag{39}$$

$$a_1 = \frac{[\sigma_1^2 - (\theta_1 - \theta_x)^2][(\theta_1 + \theta_x)^2 - \sigma_1^2]}{16\sigma_1^3 r_1} \tag{40}$$

$$a_\infty = \frac{[\sigma_\infty^2 - (\theta_0 - \theta_1)^2][(\theta_0 + \theta_1)^2 - \sigma_\infty^2]}{16\sigma_\infty^3 r_\infty} \tag{41}$$

where:

$$r_0 = r(\theta_0, \theta_x, \theta_1, \theta_\infty;\ \sigma_0,\ p_{01},\ \ p_{x1}) \tag{42}$$

$$r_1 = r(\theta_1, \theta_x, \theta_0, \theta_\infty;\ \sigma_1,\ \ -p_{01} - p_{0x}p_{x1} + p_\infty p_x + p_0 p_1,\ \ p_{0x}) \tag{43}$$

$$r_\infty = r(\theta_0, \theta_1, \theta_x, \theta_\infty;\ \sigma_\infty,\ \ -p_{0x} - p_{01}p_{x1} + p_\infty p_1 + p_0 p_x,\ \ p_{x1}) \tag{44}$$

and $r = r(...)$ is the function (32). The higher orders $\delta_0(x)$, $\delta_1(1-x)$, $\delta_\infty(x^{-1})$ depends on their arguments as (21), with coefficients which are certain rational functions of σ_1, r_1 and σ_∞, r_∞ respectively.

PROOF: The first behavior in (37) is (20). Second and third behaviors in (37) are obtained applying σ_{01} and σ_{x1} to (20). We obtain $y'(x')$, x' and then we drop the index $'$ (prime).

Formula (39) is (22). Formula (42) is (32), derived in [21] [1] [15] . To obtain (40), (43) and (41), (44), we substitute $\theta_\nu = \theta_\nu(\theta_0', \theta_x', \theta_1', \theta_\infty')$ and $p_{ij} = p_{ij}(p_{0x}', p_{01}', p_{x1}', p_0', p_x', p_1', p_\infty')$ according to (35) and (36) respectively. After re-expressing the θ's and p's as functions of the θ''s and p''s, we drop the index $'$, because *the monodromy data are the same for the given $y(x)$.* $\Box$

Remark: The parametrization of a_0 and σ_0 of a branch $y(x)$ in terms of monodromy data θ_μ, p_{ij} is obtained from the associated Fuchsian system for x small and $|\arg x| < \varphi \leq$, $0 < \varphi \leq \pi$. The basis of loops Γ of figure 1 has been chosen, which produces the specific branch. The parametrizations for a_1 σ_1 and a_∞, σ_∞ are obtained from (35) (36). As pointed out in the Remark following (35) and (36), they are the parametrization of a paricular branch around $x = 1$ ($|\arg(1-x)| < \varphi$) and $x = \infty$. These are branches of the transcendent whose branch at $x = 0$ has parameters a_0, σ_0. The other branches are otained by the action of the braid group generated by (14) and (15).

5.4 Parametrization Formulae when $\Re\sigma_i = 0$, $p_{ij} > 2$

Proposition 5 *Let PVI be give, namely $\theta_0, \theta_x, \theta_1, \theta_\infty$ are given. Let the basis Γ of figure 1 be chosen, so that the monodromy data are refered to Γ. If the monodromy data are such that $p_{0x} > 2$, $p_{x1} > 2$, $p_{01} > 1$, there exists a transcendent with branches having behavior:*

$$y(x) = x\left\{-2iA_0 \sin^2\left(i\frac{\sigma_0}{2}\ln x + \frac{\phi_0}{2} - \frac{\pi}{4}\right) + iA_0 + B_0 + \delta_0^*(x)\right\}, \quad x \to 0$$

$$y(x) = 1-(1-x)\left\{-2iA_1 \sin^2\left(i\frac{\sigma_1}{2}\ln(1-x) + \frac{\phi_1}{2} - \frac{\pi}{4}\right) + iA_1 + B_1 + \delta_1^*(x)\right\}, \quad x \to 1$$

$$y(x) = -2iA_\infty \sin^2\left(-i\frac{\sigma_\infty}{2}\ln x + \frac{\phi_\infty}{2} - \frac{\pi}{4}\right) + iA_\infty + B_\infty + \delta_\infty^*(x^{-1}), \quad x \to \infty$$

The branches are identified by the parametrization:

$$2\cos(\pi\sigma_0) = p_{0x}, \quad 2\cos(\pi\sigma_1) = p_{x1}, \quad 2\cos(\pi\sigma_\infty) = p_{01}.$$

$$B_0 = \frac{\theta_0^2 - \theta_x^2 + \sigma_0^2}{2\sigma_0^2}, \quad A = \sqrt{\frac{\theta_0^2}{\sigma_0^2} - B_0^2}, \quad \phi_0 = i\ln\frac{2r_0}{\sigma_0 A_0}$$

$$B_1 = \frac{\theta_1^2 - \theta_x^2 + \sigma_1^2}{2\sigma_1^2}, \quad A = \sqrt{\frac{\theta_1^2}{\sigma_1^2} - B_1^2}, \quad \phi_0 = i \ln \frac{2r_1}{\sigma_1 A_1}$$

$$B_\infty = \frac{\theta_0^2 - \theta_1^2 + \sigma_\infty^2}{2\sigma_\infty^2}, \quad A = \sqrt{\frac{\theta_0^2}{\sigma_\infty^2} - B_\infty^2}, \quad \phi_\infty = i \ln \frac{2r_\infty}{\sigma_\infty A_\infty}$$

$$r_0,\ r_1,\ r_\infty \text{ as in (42), (43), (44).}$$

$\delta_i^*(..)$ have the functional form of (26).

PROOF: The behavior when $x \to 0$ is (27). r_0 is (32). The behaviors at $x \to 1$ and $x \to \infty$, and the functional dependence of the integration constants on the monodromy data are proved as for (37), (40), (41), (43), (44) starting from (27) and (32), via σ_{01} and σ_{x1}. $\square$

5.5 Parametrization Formulae when $\Re\sigma_i = 1$, $p_{ij} < -2$

Proposition 6 *Let PVI be give, namely $\theta_0, \theta_x, \theta_1, \theta_\infty$ are given. Let the basis Γ of figure 1 be chosen, so that the monodromy data are refered to Γ. If $p_{0x} < -2$, $p_{x1} < -2$, $p_{01} < -2$, there exists a transcendent whith branches having behavior:*

$$y(x) = \left\{ -2iA_0 \sin^2\left(i\frac{1-\sigma_0}{2} \ln x + \frac{\phi_0}{2} - \frac{\pi}{4} \right) \ + iA_0 + B_0 \ + \delta_0^*(x) \right\}^{-1}, \quad x \to 0$$

$$y(x) = 1 - \left\{ -2iA_1 \sin^2\left(i\frac{1-\sigma_1}{2} \ln(1-x) + \frac{\phi_1}{2} - \frac{\pi}{4} \right) \ + iA_1 + B_1 \ + \delta_1^*(1-x) \right\}^{-1}, \quad x \to 1$$

$$y(x) = x \left\{ -2iA_\infty \sin^2\left(i\frac{\sigma_\infty - 1}{2} \ln x + \frac{\phi_\infty}{2} - \frac{\pi}{4} \right) \ + iA_\infty + B_\infty \ + \delta_0^*(x^{-1}) \right\}^{-1} \quad x \to \infty$$

The branches are identified by the parametrization:

$$2\cos(\pi\sigma_0) = p_{0x}, \quad 2\cos(\pi\sigma_1) = p_{x1}, \quad 2\cos(\pi\sigma_\infty) = p_{01}.$$

$$\sigma_j = 1 + i\nu_j, \quad \nu_j \in \mathbf{R}, \quad j = 0, 1, \infty$$

$$B_0 = \frac{\nu_0^2 + \theta_1^2 - (\theta_\infty - 1)^2}{2\nu_0^2}, \quad A_0 = i\sqrt{\frac{(\theta_\infty - 1)^2}{\nu_0^2} + B_0^2}, \quad \phi_0 = i \ln \frac{r_0}{(1-\sigma_0)A_0}$$

$$B_1 = \frac{\nu_1^2 + \theta_0^2 - (\theta_\infty - 1)^2}{2\nu_1^2}, \quad A_1 = i\sqrt{\frac{(\theta_\infty - 1)^2}{\nu_1^2} + B_1^2}, \quad \phi_1 = i \ln \frac{r_1}{(1-\sigma_1)A_1}$$

$$B_\infty=\frac{\nu_\infty^2+\theta_x^2-(\theta_\infty-1)^2}{2\nu_\infty^2}, \quad A_\infty=i\sqrt{\frac{(\theta_\infty-1)^2}{\nu_\infty^2}+B_\infty^2}, \quad \phi_\infty=i\ln\frac{r_\infty}{(1-\sigma_\infty)A_\infty}$$

$$r_0=\mathcal{R}(\theta_0,\theta_x,\theta_1,\theta_\infty;\sigma_0,p_{01},p_{x1})$$
$$r_1=\mathcal{R}(\theta_1,\theta_x,\theta_0,\theta_\infty;\sigma_1,-p_{01}-p_{0x}p_{x1}+p_\infty p_x+p_0p_1,\ p_{0x})$$
$$r_\infty=\mathcal{R}(\theta_0,\theta_1,\theta_x,\theta_\infty;\sigma_\infty,-p_{0x}-p_{01}p_{x1}+p_\infty p_1+p_0p_x,\ p_{x1})$$

$$\mathcal{R}(\theta_0,\theta_x,\theta_1,\theta_\infty;\sigma,p_{01},p_{x1})=\frac{(\theta_\infty-\theta_1-\sigma)(\theta_\infty+\theta_1-2+\sigma)(\theta_0+\theta_x+\sigma)}{4(1-\sigma)(\theta_0+\theta_x+2-\sigma)}\ \frac{1}{\mathbf{F}^*}, \qquad (45)$$

and

$$\mathbf{F}^*:=\frac{\Gamma(2-\sigma)^2\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma)\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma)+1\right)}{\Gamma(\sigma)^2\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma)+2\right)}\times$$
$$\times\frac{\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma)\right)}{\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma)+2\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma)+1\right)}\ \frac{\mathcal{V}}{\mathcal{U}},$$

and:

$$\mathcal{U}:=-e^{-i\pi\sigma}\left[\frac{i}{2}\sin(\pi\sigma)p_{1x}+\cos(\pi\theta_x)\cos(\pi\theta_\infty)+\cos(\pi\theta_0)\cos(\pi\theta_1)\right]\ +$$
$$-\frac{i}{2}\sin(\pi\sigma)p_{01}+\cos(\pi\theta_x)\cos(\pi\theta_1)+\cos(\pi\theta_\infty)\cos(\pi\theta_0)$$

$$\mathcal{V}:=4\sin\frac{\pi}{2}(\theta_0+\theta_x+\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_x-\sigma)\ \sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma).$$

$\delta_0^*(x)$ *is (18) and* $\delta_1^*(1-x)$, $\delta_\infty^*(x^{-1})$ *have the same functional dependence in* $1-x$ *and* x^{-1} *respectively.*

PROOF: The behavior when $x\to 0$ is (19), which is derived in section 4. In that section, (19) is obtained from (27) through the fractional linear transformation (30). Therefore $\mathcal{R}$ is obtained from r in (32), by substituting $\theta=\theta(\theta')$, $p=p(p')$ according to (30), and $\sigma=1-\sigma'$ according to (31) . After substitution, the index $'$ is dropped.

As in the proof of (37), (40), (41), (43), (44), the formulae at $x=1,\infty$ are obtained applying the fractional linear transformations σ_{01} and σ_{x1} to the formulae of the behavior at $x=0$. $\square$

Proposition 7 *The inverse formula of (45), analogous to (33), is as follows. Let*

$$\mathbf{G}_i=\mathbf{G}_i(\theta_0,\theta_x,\theta_1,\theta_\infty,\sigma), \quad i=1,2,3,4,5,6$$

indicate the functional dependence of the coefficients $\mathbf{G}_i$ *in (33). Then, let us define:*

$$\mathbf{G}_i^*=\mathbf{G}_i(\theta_\infty-1,\theta_1,\theta_x,\theta_0+1,1-\sigma).$$

Then, the analogous of (33) for the case $\Re\sigma = 1$ is:

$$\begin{cases} p_{0x} & = & 2\cos\pi\sigma \\ \\ p_{x1} & = & \mathbf{G}_1^*\, r^{-1} + \mathbf{G}_2^* + \mathbf{G}_3^*\, r \\ \\ p_{01} & = & -\mathbf{G}_4^*\, r^{-1} \; - \; \mathbf{G}_5^* \; - \; \mathbf{G}_6^*\, r \end{cases} \tag{46}$$

PROOF: Application of (30). $\square$

5.6 Parametrization Formulae in general

The above propositions assume that p_{0x}, p_{x1}, p_{01} are of the same tipe (namely, for example, all greater than 2 or smaller than -2) The mixed case is of course the one to expect, for example $p_{0x} < -2$, $p_{01} \notin (-\infty, -2] \cup [2, \infty)$ and $p_{01} > 2$. Any other combination of values of the $p_{ij}'s$ is possible. Therefore:

Let PVI be give, namely $\theta_0, \theta_x, \theta_1, \theta_\infty$ are given. Let the basis Γ of figure 1 be chosen, so that the monodromy data are refered to Γ. For the given monodromy data, there exists a transcendent with branch having the behavior of Proposition 4 at $x = 0$ if $p_{0x} \notin (-\infty, -2] \cup [2, \infty)$, of Proposition 5 if $p_{0x} > 2$, and of Proposition 6 if $p_{0x} < -2$. At $x = 1$ the branch has the behavior of Proposition 4 if $p_{x1} \notin (-\infty, -2] \cup [2, \infty)$, of Proposition 5 if $p_{x1} > 2$, and of Proposition 6 if $p_{x1} < -2$. At $x = \infty$ the branch has the behavior of Proposition 4 if $p_{01} \notin (-\infty, -2] \cup [2, \infty)$, of Proposition 5 if $p_{01} > 2$, and of Proposition 6 if $p_{01} < -2$. The branch is identified by the corresponding parametrizations given at $x = 0, 1, \infty$ respectively in Propositions 4, 5, 6.

5.7 Solution of the Connection Problem

Suppose that we know the behavior of $y(x)$ at the critical points $x = 0$. We want to write the behavior at the other critical points.

– From the given behavior, we extract the exponent σ_0 and a_0 (or ϕ_0). From a_0 (or ϕ_0) we compute r_0.

– Given σ_0 and r_0, we can compute p_{0x}, p_{x1}, p_{01} from formulae (33) or (46) (where $r = r_0$, $\sigma = \sigma_0$).

– Then, we substitute p_{0x}, p_{x1}, p_{01} in formulae of proposition 4, 5, or 6 and we obtain a_1 (or ϕ_1) and σ_1, a_∞ (or ϕ_∞) and σ_∞.

6 PVI associated to a Frobenius manifolds

The structure of a semi-simple Frobenius manifold of dimension 3 is described by a solution of a PVI equation with $\beta = \gamma = 0$, $\delta = \frac{1}{2}$, which means $\theta_0 = \theta_x = \theta_1 = 0$ [8]. For the solutions (20) and (27), namely $0 \leq \Re\sigma < 1$, r reduces to:

$$r = r(0, 0, 0, \theta_\infty, \sigma, p_{01}, p_{x1})$$

$$= \frac{\sigma\ \mathcal{G}^2(\sigma,\theta_\infty)\mathcal{F}^2(\sigma,\theta_\infty)}{\sin^2\pi\sigma}\Big[(1+\cos\pi\theta_\infty)(1-e^{i\pi\sigma})+\frac{i}{2}\sin\pi\sigma\ (p_{01}+p_{x1}e^{i\pi\sigma})\Big]$$

where

$$\mathcal{G}(\sigma,\theta_\infty)=\frac{4^{-\sigma}\Gamma\left(\frac{1-\sigma}{2}\right)^2}{\Gamma\left(1-\frac{\theta_\infty}{2}-\frac{\sigma}{2}\right)\Gamma\left(\frac{\theta_\infty}{2}-\frac{\sigma}{2}\right)},\quad \mathcal{F}(\sigma,\theta_\infty)=\frac{\cos^2\left(\frac{\pi}{2}\sigma\right)}{\cos\pi\sigma-\cos\pi\theta_\infty}$$

We remark that the above formulas hold true if $\sigma\neq 0,\pm\theta_\infty+2m$, $m\in\mathbf{Z}$. In [14] we computed r *for every case when* $\sigma\notin(-\infty,0)\cup[1,+\infty)$ and $\theta_\infty\neq 0$. Please, refer to [14], page 298-301, Theorem 2.

The connection problem is solved as in the general case

$\diamond$ We now consider a solution with $\Re\sigma_0=1$, namely $p_{0x}<-2$. This special case is derived in subsection 4.1. The critical behavior is:

$$y(x)=\left\{1-\frac{\nu_0^2+(\theta_\infty-1)^2}{\nu_0^2}\sin^2\left(\frac{\nu_0}{2}\ln x+\frac{\phi_0}{2}-\frac{\pi}{4}\right)\ +\delta_0^*(x)\right\}^{-1},\quad x\to 0.$$

$$2\cos\pi\sigma_0=p_{0x}<-2,\quad \sigma_0=1+i\nu_0,\quad \phi_0=i\ln\left(\frac{4r_0\nu_0}{\nu_0^2+(\theta_\infty-1)^2}\right)+\pi$$

$$r_0=\mathcal{R}(0,0,0,\theta_\infty,\sigma_0,p_{01},p_{x1})=$$

$$=\frac{16^{\sigma_0}\Gamma^2\left(1+\frac{1}{2}(\theta_\infty-\sigma_0)\right)\Gamma^2\left(2-\frac{1}{2}(\theta_\infty+\sigma_0)\right)}{4(1-\sigma_0)^3\ (\sin\pi\sigma_0)^2\ \Gamma^4\left(\frac{1-\sigma_0}{2}\right)}\times$$

$$\times\ \left[(1+\cos\pi\theta_\infty)(1-e^{-i\pi\sigma_0})-\frac{i}{2}\sin\pi\sigma_0\ (p_{01}+p_{x1}e^{-i\pi\sigma_0})\right]$$

If also $p_{x1}<-2$, then $y(x)$ has behavior:

$$y(x)=1-\left\{1-\frac{\nu_1^2+(\theta_\infty-1)^2}{\nu_1^2}\sin^2\left(\frac{\nu_1}{2}\ln(1-x)+\frac{\phi_1}{2}-\frac{\pi}{4}\right)\ +\delta_1^*(1-x)\right\}^{-1},\quad x\to 1.$$

$$2\cos\pi\sigma_1=p_{x1}<-2,\quad \sigma_1=1+i\nu_1,\quad \phi_1=i\ln\left(\frac{4r_1\nu_1}{\nu_1^2+(\theta_\infty-1)^2}\right)+\pi$$

$$r_1=\mathcal{R}(0,0,0,\theta_\infty;\sigma_1,-p_{01}-p_{0x}p_{x1}+4(\cos(\pi\theta_\infty)+1),p_{0x})=$$

$$=\frac{16^{\sigma_1}\Gamma^2\left(1+\frac{1}{2}(\theta_\infty-\sigma_1)\right)\Gamma^2\left(2-\frac{1}{2}(\theta_\infty+\sigma_1)\right)}{4(1-\sigma_1)^3\ (\sin\pi\sigma_1)^2\ \Gamma^4\left(\frac{1-\sigma_1}{2}\right)}\ \times$$

$$\times\ \left[(1+\cos\pi\theta_\infty)(1-e^{i\pi\sigma_1})-\frac{i}{2}\sin\pi\sigma_1\ (p_{0x}e^{-i\pi\sigma_1}-p_{01}-p_{0x}p_{x1})\right]$$

If also $p_{01} < -2$, then $y(x)$ has behavior:

$$y(x) = x\left\{1 - \frac{\nu_\infty^2 + (\theta_\infty - 1)^2}{\nu_\infty^2} \sin^2\left(-\frac{\nu_\infty}{2}\ln x + \frac{\phi_\infty}{2} - \frac{\pi}{4}\right) + \delta_\infty^*(1-x)\right\}^{-1}, \quad x \to \infty.$$

$$2\cos\pi\sigma_\infty = p_{01} < -2, \quad \sigma_\infty = 1 + i\nu_\infty, \quad \phi_\infty = i\ln\left(\frac{4r_\infty\nu_\infty}{\nu_\infty^2 + (\theta_\infty - 1)^2}\right) + \pi$$

$$r_\infty = \mathcal{R}(0,0,0,\theta_\infty;\sigma_\infty, -p_{0x} - p_{01}p_{x1} + 4(\cos(\pi\theta_\infty)+1), p_{x1}) =$$

$$= \frac{16^{\sigma_\infty}\Gamma^2\left(1 + \frac{1}{2}(\theta_\infty - \sigma_\infty)\right)\Gamma^2\left(2 - \frac{1}{2}(\theta_\infty + \sigma_\infty)\right)}{4(1-\sigma_\infty)^3\ (\sin\pi\sigma_\infty)^2\ \Gamma^4\left(\frac{1-\sigma_\infty}{2}\right)} \times$$

$$\times\left[(1 + \cos\pi\theta_\infty)(1 - e^{i\pi\sigma_\infty}) - \frac{i}{2}\sin\pi\sigma_\infty\ (p_{x1}e^{-i\pi\sigma_\infty} - p_{0x} - p_{01}p_{x1})\right]$$

7 The Full Expansion

The full asymptotic expansion of a solution $y(x)$ when $x \to 0$, for $0 \le \Re\sigma < 1$, is:

$$\boxed{y(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n}^{n} c_{nm} x^{m\sigma}, \quad 0 \le \Re\sigma < 1.} \tag{47}$$

The above series can be rigorously obtained from the elliptic representation of PVI of [15]. This is explained in Appendix II, where the series follows form $\delta_E(x)$ and $\delta_E^*(x)$, $\phi(x)$ in (60) and (61) respectively (where $\nu_2 = 1 - \sigma$. A special attention must be payed for the case of $\delta_E^*(x)$, $\phi(x)$. See section 9.3). In [15] the author proved the convergence of $y(x)$ by solving some integral equations with successive approximations, derived from the elliptic representation of PVI (PVI is written as a system of two first order equations, which are solved by their associated integral equations. The solution is constructed as a series by successive approximations. A similar procedure was first introduced by S.Shimomura (review in [20])).

On the other hand, the proof of [15] does not fix the bound $-n \le m \le n$ (i.e. the upper bound of m_1 in (60), (61) must be $2m_1 + 1$, but it is not determined by the procedure of [15]), and it gives no recursive procedure to compute the coefficients c_{nm}. This is instead possible by the recursive computational procedure explained below, by substitution of the above series into PVI. All the coefficients are determined recursively in terms of σ and another parameter r. The series (47) gives the series of $\delta(x)$ and $\delta^*(x)$ in Propositions 2 and 3.

In order to compute the coefficients c_{nm}, we write PVI as $Eq = 0$, where:

$$Eq := -\frac{d^2y}{dx^2} + \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right]$$

Let us substitute the expansion (47) into $Eq=0$. We observe that:

$$Eq=\frac{\text{numerator}}{\text{denominator}}, \qquad \text{denominator}=2y(1-y)(y-x)x^2(x-1)^2$$

The denominator is not zero for $x\neq 0,1,\infty$ and $y\neq 0,x,1$. So, the coefficients are determined by

$$\text{numerator}=0$$

Let c denote the c_{nm}'s. The explicit computation gives:

$$\text{numerator}=\xi_3(x,c)x^3+\xi_4(x,c)x^4+\xi_5(x,c)x^5+...=\sum_{l=3}^{\infty}\xi_l(x,c)x^l.$$

where $\xi_l(x,c)$ depends on $c_{l-2,l-2}$, $c_{l-2,l-3}$, ..., $c_{l-2,2-l}$ and on c_{km}, $k\leq l-3$. The first term $\xi_3(x,c)$ is:

$$\xi_3(x,c)=\sum_{k=-2}^{2}\xi_{3k}(c)x^{k\sigma}$$

where:

$$\xi_{32}(c)=\xi_{32}(c_{10},c_{11}), \quad \xi_{3k}(c)=\xi_{3k}(c_{10},c_{11},c_{1,-1}), \quad k=1,0,-1,-2.$$

We choose c_{11} to be the free parameter (integration constant, the other being σ). The coefficients

$$\xi_{32}(c)=\xi_{32}(c_{10},c_{11}), \quad \xi_{31}(c)=\xi_{31}(c_{10},c_{11},c_{1,-1})$$

are linear in $c_{10},c_{1,-1}$. Then,

$$\xi_{32}(c_{10},c_{11})=0 \text{ determines } c_{10}$$

Substitute c_{10} into $\xi_{31}(c)$. Then:

$$\xi_{31}(c_{10},c_{11},c_{1,-1})=0 \text{ determines } c_{1,-1}$$

For example, if we write:

$$c_{11}=-\frac{r}{\sigma},$$

where r is a new free parameter, we find:

$$c_{10}=B, \quad c_{1,-1}=\frac{\sigma A^2}{4r},$$

where:

$$B=\frac{\sigma^2-2\beta-1+2\delta}{2\sigma^2}, \quad A^2+B^2=-\frac{2\beta}{\sigma^2}.$$

If now we substitute $c_{10},c_{1,-1}$ in

$$\xi_{30}(c_{10},c_{11},c_{1,-1}), \quad \xi_{3,-1}(c_{10},c_{11},c_{1,-1}), \quad \xi_{3,-2}(c_{10},c_{11},c_{1,-1})$$

we verify that they vanish. Namely $\xi_3(x,c)=0$.

The next step is to solve

$$\xi_4(x,c)=0$$

First, we substitute into $\xi_4(x,c)$ the integration constant c_{11} and the coefficients $c_{10}, c_{1,-1}$ obtained in the previous step. We find:

$$\xi_4(x,c)=\sum_{k=-4}^{4}\xi_{4k}(c)x^{k\sigma}$$

where the ξ_{4k}'s are linear in c_{2m}. Precisely:

$$\xi_{44}(c_{2,-2})=0 \quad \text{determines} \quad c_{2,-2}$$

$$\xi_{43}(c_{2,-2},c_{2,-1})=0 \quad \text{determines} \quad c_{2,-1}$$

$$\xi_{42}(c_{2,-2},c_{2,-1},c_{20})=0 \quad \text{determines} \quad c_{20}$$

$$\xi_{41}(c_{2,-2},c_{2,-1},c_{20},c_{21})=0 \quad \text{determines} \quad c_{21}$$

$$\xi_{40}(c_{2,-2},c_{2,-1},c_{20},c_{21},c_{22})=0 \quad \text{determines} \quad c_{22}$$

Substituting the above solutions into $\xi_4(x,c)$, we find $\xi_4(x,c)=0$, namely $\xi_{4,-1},\xi_{4,-2},\xi_{4,-3},\xi_{4,-4}$ vanish on the above solutions c_{2m}.

If we proceed with $\xi_5=0$ we find again

$$\xi_5(x,c)=\sum_{k=-5}^{5}\xi_{5k}(c)x^{k\sigma}$$

The coefficients ξ_{5k} are linear in c_{3m}. We solve

$$\xi_{5k}=0, \quad k=5,4,3,2,1,0,-1$$

and determine uniquely

$$c_{33},\ c_{32},\ c_{31},\ c_{30},\ c_{3,-1},\ c_{3,-2},\ c_{3,-3}$$

respectively. $\xi(x,c)$ vanishes identically on these solutions.

This is a recursive procedure to determine $y(x)$ at all orders x^n. In general,

$$\xi_n(x,c)=\sum_{k=-n}^{n}\xi_{nk}(c)x^{k\sigma}=0$$

determines $c_{n-2,n-2}, c_{n-2,n-3}, ..., c_{n-2,-n+2}$ uniquely. The crucial point is that, for any n, $\xi_{nk}(c)$, $k=n,n-1,...,4-n$ is *linear* in $c_{n-2,m}$ and we have a *finite* number of terms $x^{k\sigma}$, $k=-n,...,n$.

We can extract the **leading term** of the expansion when $0\le\Re\sigma<1$, to check that the result is in accordance with propositions 2 and 3:

– If $0 < \Re\sigma < 1$:

$$y(x) = ax^{1-\sigma}(1+\delta(x)), \qquad \delta(x) = -1 + \sum_{n=0}^{\infty} x^n \sum_{m=-n}^{n+2} \tilde{c}_{nm} x^{m\sigma}$$

$$\tilde{c}_{nm} = \frac{c_{n+1,m-1}}{a}, \quad a = c_{1,-1} = \frac{\sigma A^2}{4r}.$$

Note that $\delta(x) \to 0$ as $x \to 0$. We can also write

$$\delta(x) = \sum_{m_2=0}^{\infty} \sum_{m_1=0}^{2m_2+2} \delta_{m_1 m_2} x^{m_1\sigma} x^{m_2(1-\sigma)}, \qquad m_1 + m_2 \geq 1, \quad \delta_{m_1 m_2} = \tilde{c}_{m_2, m_1-m_2}$$

– If $\Re\sigma = 0$:

$$y(x) = x\left\{-2iA\sin^2\left(i\frac{\sigma}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) + iA + B + \delta(x)\right\},$$

$$\phi = i\ln\frac{2r}{\sigma A},$$

$$\delta(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n-1}^{n+1} b_{nm} x^{m\sigma} = \sum_{m_2=1}^{\infty} \sum_{m_1=-1}^{2m_2+1} a_{m_1 m_2} x^{m_1\sigma} x^{m_2(1-\sigma)},$$

$$b_{nm} = c_{n+1,m}, \quad a_{m_1 m_2} = b_{m_2, m_1-m_2}.$$

Remarks:

1) The computation of $y(x)$ can be done without assumptions on σ. The only condition is $\sigma \notin \mathbf{Z}$, to avoid vanishing denominators in c_{nm}. If we assume $-1 < \Re\sigma < 1$, the expansions are convergent for small x and $\arg x$ bounded. If we further assume $0 \leq \Re\sigma < 1$ the leading term is as above.

2) Observe that $c_{nm} \sim r^m$. Moreover, observe that he coefficients c_{nm} with negative m contain the factor $(\sigma^2-(\theta_0+\theta_x)^2)(\sigma^2-(\theta_0-\theta_x)^2)$. Thus, if $\sigma \in \{\pm(\theta_0+\theta_x), \pm(\theta_0-\theta_x)\}$, these c_{nm} vanish, and we have:

$$y(x) = \sum_{n=1}^{\infty} x^n \sum_{m=0}^{n} c_{nm} x^{m\sigma} = \sum_{N=0}^{\infty} y_N(x)\ (r\ x^{\sigma})^N, \qquad \sigma \in \{\pm(\theta_0+\theta_x), \pm(\theta_0-\theta_x)\}, \quad (48)$$

where $y_N(x)$ are Taylor expansions of the form:

$$\begin{array}{lcccccccc}
y_0(x) = & y_1^0 x & + & y_2^0 x^2 & + & y_3^0 x^3 & + & y_4^0 x^4 & + \ \ldots \\
y_1(x) = & y_1^1 x & + & y_2^1 x^2 & + & y_3^1 x^3 & + & y_4^1 x^4 & + \ \ldots \\
y_2(x) = & & & y_2^2 x^2 & + & y_3^3 x^3 & + & y_4^2 x^4 & + \ \ldots \\
\vdots & & & & & & & & \\
y_N(x) = & & & & & y_N^N x^N & + & y_{N+1}^N x^{N+1} & + \ \ldots
\end{array}$$

The condition that $|x^{1+\sigma}|$ is the dominant term (namely $|x^{1+\sigma}| > |x^{n+m\sigma}|$, $\forall n \geq 1, 0 \leq m \leq n$) is: $-1 < \Re\sigma < 0$. The condition that $|x|$ is the dominant term (namely $|x| > |x^{n+m\sigma}|$, $\forall n \geq 1, 0 \leq m \leq n$) is: $\Re\sigma > 0$. The condition that $|x^{1+\sigma}|$ is greater than $|x^n|$, $\forall n \geq 2$ is: $\Re\sigma < 1$. Therefore, if $-1 < \Re\sigma < 1$, $\sigma \neq 0$, the leading terms of (48) are (23) and (24), namely:

$$y(x) = \frac{\theta_0}{\theta_0+\theta_x}\, x \;\mp\; \frac{r}{\theta_0+\theta_x}\, x^{1+\sigma} + ..., \quad \sigma = \pm(\theta_0+\theta_x) \neq 0,$$
$$y(x) = \frac{\theta_0}{\theta_0-\theta_x}\, x \;\mp\; \frac{r}{\theta_0-\theta_x}\, x^{1+\sigma} + ..., \quad \sigma = \pm(\theta_0-\theta_x) \neq 0.$$

The higher order terms are a convergent expansion. We observe that formula (32) has limit when σ tends to $\pm(\theta_0+\theta_x)$, $\pm(\theta_0-\theta_x)$, so it applies here as well. If moreover the above σ is also purely immaginary ($\sigma = i\nu$, $\nu \in \mathbf{R}\backslash\{0\}$), the above expansions become (28) and (29).

If $\Re\sigma \leq -1$, no convergence is expected. The expansion (48) is proved to be convergent for $-1 < \Re\sigma < 1$. We expect (but not prove here) to be convergent also for any positive $\Re\sigma$. The inequality $|x^K| > |x^{1+\sigma}|$ holds for $\Re\sigma > K-1$, $K \geq 1$ integer. Therefore, from (48), one deduces that PVI has for positive $\Re\sigma$ two out of the four solutions of the form:

$$y(x) = \frac{\theta_0}{\theta_0+\theta_x}\, x \;+ \sum_{n=2}^{K} y_n^0 x^n \;\mp\; \frac{r}{\theta_0+\theta_x}\, x^{1+\sigma} + ..., \quad \sigma = \pm(\theta_0+\theta_x) \neq 0,$$

$$y(x) = \frac{\theta_0}{\theta_0-\theta_x}\, x \;+ \sum_{n=2}^{K} y_n^0 x^n \;\mp\; \frac{r}{\theta_0-\theta_x}\, x^{1+\sigma} + ..., \quad \sigma = \pm(\theta_0-\theta_x) \neq 0.$$

The integration constant r appears in the $K+1 = [|\Re(\theta_0 \pm \theta_x)|] + 2$ term.[5]

$\diamond$ The asymptotic expansion for $\Re\sigma = 1$ is obtained from (47) through (30), with the substitution $\sigma \mapsto 1-\sigma$ (see section 4):

$$\boxed{y(x)^{-1} = \sum_{n=0}^{\infty} x^n \sum_{m=-n-1}^{n+1} d_{nm}\; x^{m(1-\sigma)}, \quad \Re\sigma = 1.}$$

Practically, to compute the coefficients d_{nm}, let us call the above solution $y'(x)$, the exponent σ' and the parameters θ'_μ. Then, we compute the coefficients of $y(x)$, the image

[5] Observe also that we can choose $c_{1,-1}$ as integration constant, instead of c_{11}. Say that we put $c_{1,-1} = \tilde{r} \in \mathbf{C}$. We find that $c_{nm} \sim \tilde{r}^{-m}$. This time, the c_{nm} with positive m have factors $(\sigma^2 - (\theta_0 + \theta_x)^2)(\sigma^2 - (\theta_0 - \theta_x)^2)$, again leading to:

$$y(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n}^{0} c_{nm} x^{m\sigma} = \sum_{N=0}^{\infty} \tilde{y}_N(x)\; (\tilde{r}\; x^{-\sigma})^N \quad \text{if } \sigma \in \{\pm(\theta_0+\theta_x), \pm(\theta_0-\theta_x)\}$$

This is again (48).

of $y'(x)$ via (30), with $\sigma = 1-\sigma'$ ($\Re\sigma = 0$). Let c_{nm} be the coefficents of the $y(x)$ in (47). Then, we have:

$$\Big(y'(x)\Big)^{-1} = \frac{1}{x}y(x) = \frac{\sum_{n=1}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}x^{m\sigma}}{x} = \sum_{n=0}^{\infty} x^n \sum_{m=-n-1}^{n+1} c_{n+1,m}\ x^{m\sigma}.$$

This proves that:

$$d_{nm} = c_{n+1,m}$$

We extract the leading terms. Dropping again the index $'$, the final result when $\Re\sigma = 1$ is then in accordance with proposition 1:

$$y(x)^{-1} = \frac{r}{\sigma-1}x^{1-\sigma} + B + \frac{(1-\sigma)A^2}{4r}x^{\sigma-1} + \delta^*(x)$$

$$= -2iA\sin^2\left(i\frac{1-\sigma}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) + iA + B + \delta^*(x), \quad \Re\sigma = 1,$$

where:

$$B = \frac{(1-\sigma)^2 - 2\gamma + 2\alpha}{2(1-\sigma)^2}, \quad A^2 + B^2 = \frac{2\alpha}{(1-\sigma)^2}, \quad \phi = i\ln\frac{2r}{(1-\sigma)A}$$

and

$$\delta^*(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n-1}^{n+1} d_{nm}\ x^{m(1-\sigma)} = \sum_{m_1=1}^{\infty}\sum_{m_2=-1}^{2m_1+1} e_{m_1m_2}x^{m_1\sigma}x^{m_2(1-\sigma)},$$

$e_{m_1m_2} = d_{m_1,m_2-m_1}$.

Note: The full expansion for the logarithmic solutions can be obtained by substituting into PVI the following:

$$y(x) = x(A_1 + B_1\ln x + C_1\ln^2 x + D_1\ln^3 x + ...) + x^2(A_2 + B_2\ln x + ...) + ..., \qquad x \to 0.$$

We obtain:

$$y(x) = \begin{cases} \frac{\theta_0}{\theta_0 \pm \theta_x}x + O(x^2) \quad \text{[Taylor expansion]}, \\ x\ \left(\frac{\theta_0^2 - B_1^2}{\theta_0^2-\theta_x^2} + B_1\ln x + \frac{\theta_x^2-\theta_0^2}{4}\ln^2 x\right) + x^2(...) + ..., \\ x\ (A_1 \pm \theta_0\ln x) + x^2(...) + ..., \qquad \text{and } \theta_0 = \pm\theta_x. \end{cases}$$

A_1 and B_1 are parameters. The other expansions are ontained applying the symmetries to the above.

7.1 Full Expansions at $x = 1, \infty$

If the three exponents σ_0, σ_1, σ_∞ satisfy

$$0 \le \Re\sigma_i < 1, \quad i = 0, 1, \infty$$

the full expansion for $y(x)$ at the three critical points can be computed with the symmetries σ_{01} and σ_{x1} of section 5.

$$y(x) = \begin{cases} \sum_{n=1}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}^{(0)} \; x^{m\sigma_0}, \quad x \to 0 \\ 1 - \sum_{n=1}^{\infty}(1-x)^n \sum_{m=-n}^{n} c_{nm}^{(1)} \; (1-x)^{m\sigma_1}, \quad x \to 1 \\ \sum_{n=0}^{\infty} x^{-n} \sum_{m=-n-1}^{n+1} c_{nm}^{(\infty)} \; x^{-m\sigma_\infty}, \quad x \to \infty \end{cases}$$

where:

$$c_{nm}^{(0)} = c_{nm}^{(0)}(\sigma_0, \theta_0, \theta_x, \theta_1, \theta_\infty, r_0),$$

$$c_{nm}^{(1)} = c_{nm}^{(0)}(\sigma_1, \theta_1, \theta_x, \theta_0, \theta_\infty, r_1), \quad c_{nm}^{(\infty)} = c_{n+1,m}^{(0)}(\sigma_\infty, \theta_0, \theta_1, \theta_x, \theta_\infty, r_\infty)$$

See section 5 for the notations r_0, r_1, r_∞.

As we already explained, if $\Re\sigma_0 = 1$, the full expansion for $x \to 0$ is:

$$y(x) = \frac{1}{\sum_{n=0}^{\infty} x^n \sum_{m=-n-1}^{n+1} d_{nm}^{(0)} \; x^{m(1-\sigma_0)}}, \quad x \to 0,$$

$$d_{nm}^{(0)} = d_{nm}^{(0)}(\sigma_0, \theta_0, \theta_x, \theta_1, \theta_\infty, r_0).$$

If also $\Re\sigma_1 = 1$, the full expansion for $x \to 1$ is:

$$y(x) = 1 - \frac{1}{\sum_{n=0}^{\infty}(1-x)^n \sum_{m=-n-1}^{n+1} d_{nm}^{(1)} \; (1-x)^{m(1-\sigma_1)}}, \quad x \to 1,$$

$$d_{nm}^{(1)} = d_{nm}^{(0)}(\sigma_1, \theta_1, \theta_x, \theta_0, \theta_\infty, r_1).$$

If also $\Re\sigma_\infty = 1$, the full expansion for $x \to \infty$ is:

$$y(x) = \frac{x}{\sum_{n=0}^{\infty} x^{-n} \sum_{m=-n-1}^{n+1} d_{nm}^{(\infty)} \; x^{-m(1-\sigma_\infty)}}, \quad x \to \infty,$$

$$d_{nm}^{(\infty)} = d_{nm}^{(0)}(\sigma_\infty, \theta_0, \theta_1, \theta_x, \theta_\infty, r_\infty).$$

8 Appendix I: Derivation of the critical behavior when $0 \leq \Re\sigma < 1$

8.1 Critical Behavior of the Solution of the Schlesinger Equations

The critical behavior follows from the Lemma 2.4.8 at page 262 of [29], applied to the Schlesinger equations of the Fuchsian system of PVI. Let $\hat{A}_0$, $\hat{A}_x$, $\hat{A}_1$ be independent of x and satisfy the following conditions:

$$\text{Eigenvalues } \hat{A}_j = \frac{\theta_j}{2}, \quad -\frac{\theta_j}{2}, \quad j = 0, x, 1; \quad \hat{A}_0 + \hat{A}_x + \hat{A}_1 = -\frac{\theta_\infty}{2} \, \sigma_3,$$

We also observe that $\mathrm{Tr}(\hat{A}_0 + \hat{A}_x) = 0$, so the eigenvalues have opposite sign. Let them be:

$$\frac{\sigma}{2}, -\frac{\sigma}{2} \ := \ \text{eigenvalues of } \Lambda := \hat{A}_0 + \hat{A}_x.$$

$\diamond$ Computation of $\hat{A}_1$ and Λ. Suppose that $\theta_\infty \neq 0$. Let $r_1 \in \mathbf{C}$, $r_1 \neq 0$. The condition of given eigenvalues and the relation $\Lambda + \hat{A}_1 = -\frac{\theta_\infty}{2}\sigma_3$ immediately imply:

$$\hat{A}_1 = \begin{pmatrix} \frac{\sigma^2-\theta_\infty^2-\theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\sigma^2-(\theta_1-\theta_\infty)^2][\sigma^2-(\theta_1+\theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\sigma^2-\theta_\infty^2-\theta_1^2}{4\theta_\infty} \end{pmatrix}, \tag{49}$$

and

$$\Lambda = \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2-\sigma^2-\theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\sigma^2-(\theta_1-\theta_\infty)^2][\sigma^2-(\theta_1+\theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\theta_1^2-\sigma^2-\theta_\infty^2}{4\theta_\infty} \end{pmatrix}. \tag{50}$$

$\diamond$ Computation of $\hat{A}_0$ and $\hat{A}_x$. For our purposes it is enough to consider the case when $\sigma \neq 0$, so that Λ is diagonalizable (for $\sigma = 0$ see [17]). Let G_0 be the diagonalizing matrix:

$$G_0^{-1}\Lambda G_0 = \frac{\sigma}{2}\ \sigma_3, \qquad G_0 = \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty+\sigma)^2-\theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty-\sigma)^2-\theta_1^2}{4\theta_\infty r_1} \end{pmatrix}.$$

Let us denote:

$$\hat{\hat{A}}_i = G_0^{-1}\hat{A}_i G_0, \qquad i = 0, x.$$

Let $r \in \mathbf{C}$, $r \neq 0$. If $\sigma \neq 0$, we have:

$$\hat{\hat{A}}_0 = \begin{pmatrix} \frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} & r \\ -\frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r} & -\frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} \end{pmatrix}, \tag{51}$$

$$\hat{\hat{A}}_x = \begin{pmatrix} \frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} & -r \\ \frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r} & -\frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} \end{pmatrix}. \tag{52}$$

The lemma 2.4.8 at page 262 of [29], becomes the theorem at page 1145-1146 of [21], namely:

Lemma 1 *Suppose that $|\Re\sigma| < 1$. Choose two positive numbers σ_1 and K such that:*

$$|\Re\sigma| < \sigma_1 < 1, \qquad ||\hat{A}_i|| < K, \quad i = 0, x, 1.$$

Then, for every $\varphi > 0$ there exists $\epsilon > 0$ such that the Schlesinger equations have a unique solution $A_0(x), A_x(x), A_1(x)$ holomorphic in the sector $\{x \mid \ 0 < |x| < \epsilon, \ \ |arg\ x| < \varphi\}$, and satisfying the asymptotic conditions:

$$||A_1 - \hat{A}_1|| < K|x|^{1-\sigma_1}, \qquad ||x^{-\Lambda}(A_1 - \hat{A}_1)x^{\Lambda}|| < K^2|x|^{1-\sigma_1}$$

$$||x^{-\Lambda}A_0x^{\Lambda} - \hat{A}_0|| < K|x|^{1-\sigma_1}, \qquad ||x^{-\Lambda}A_xx^{\Lambda} - \hat{A}_x|| < K|x|^{1-\sigma_1}$$

Lemma 2 *The asymptotic behavior of A_1 is:*

$$A_1(x) = \hat{A}_1 + \Delta_1(x), \qquad \Delta_1(x) = O(x^{1-\sigma_1}), \quad x^{-\Lambda}\Delta_1(x)x^{\Lambda} = O(x^{1-\sigma_1}).$$

The asymptotic behaviors of A_0 and A_x are:

$$A_j(x) \ = x^{\Lambda}\hat{A}_j x^{-\Lambda} \ + \Delta_j(x) \ = G_0\Big[x^{\frac{\sigma}{2}\sigma_3}\hat{\hat{A}}_j x^{-\frac{\sigma}{2}\sigma_3}\Big]G_0^{-1} \ + \Delta_j(x),$$

$$\Delta_j(x) = O(x^{1-\sigma_1-|\Re\sigma|}), \quad j = 0, x$$

where

$$x^{\frac{\sigma}{2}\sigma_3}\hat{\hat{A}}_0 x^{-\frac{\sigma}{2}\sigma_3} = \begin{pmatrix} \frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} & r x^{\sigma} \\ -\frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r}\ x^{-\sigma} & -\frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} \end{pmatrix},$$

$$x^{\frac{\sigma}{2}\sigma_3}\hat{\hat{A}}_x x^{-\frac{\sigma}{2}\sigma_3} = \begin{pmatrix} \frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} & -r x^{\sigma} \\ \frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r}\ x^{-\sigma} & -\frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} \end{pmatrix}.$$

Proof: The behavior of A_1 is immediately obtained from lemma 1. The behaviors of A_0, A_x follow from lemma 1:

$$A_j = x^{\Lambda}\hat{A}_j x^{-\Lambda} + x^{\Lambda}\tilde{\Delta}_j x^{-\Lambda}, \quad \tilde{\Delta}_j(x) = O(x^{1-\sigma_1}), \qquad j = 0, x.$$

Observe that:

$$x^{\Lambda}\tilde{\Delta}_j x^{-\Lambda} = G_0\ x^{\frac{\sigma}{2}\sigma_3}\ (G_0^{-1}\tilde{\Delta}_j G_0)\ x^{-\frac{\sigma}{2}\sigma_3}\ G_0^{-1}.$$

Since G_0 is constant, $x^{\frac{\sigma}{2}\sigma_3}\ (G_0^{-1}\tilde{\Delta}_j G_0)\ x^{-\frac{\sigma}{2}\sigma_3}$ has form:

$$x^{\frac{\sigma}{2}\sigma_3}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} x^{-\frac{\sigma}{2}\sigma_3} = \begin{pmatrix} m_{11} & m_{12}x^{\sigma} \\ m_{21}x^{-\sigma} & m_{22} \end{pmatrix}.$$

the results follows, with $\Delta_j = x^{\Lambda}\tilde{\Delta}_j x^{-\Lambda}$. □

8.1.1 Critical Behavior of $y(x)$

As it is known, the Schlesinger equations can be written in Hamiltonian form and reduce to PVI, being the transcendent $y(x)$ solution of $A(y(x),x)_{1,2} = 0$. Namely:

$$y(x) = \frac{x\ (A_0)_{12}}{x\ [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}},$$

Lemma 2 implies:

$$(A_0)_{12} = r_1\left\{\frac{[\sigma^2-(\theta_0-\theta_x)^2][(\theta_0+\theta_x)^2-\sigma^2]}{16\sigma^3 r}\ x^{-\sigma} + \frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2} - \frac{r}{\sigma}\ x^{\sigma}\right\} \ + \delta_0(x),$$

$$(A_1)_{12} = -r_1 + \delta_1(x), \quad \delta_0(x) = O(x^{1-\sigma_1-|\Re\sigma|}), \quad \delta_1(x) = O(x^{1-\sigma_1})$$

For brevity, let us write $(A_0)_{12}=ax^{-\sigma}+bx^{\sigma}+c+\delta_0(x)$. Thus:

$$y(x)=\frac{x(ax^{-\sigma}+bx^{\sigma}+c+\delta_0(x))}{x(ax^{-\sigma}+bx^{\sigma}+c-r_1+\delta_0(x)+\delta_1(x))+r_1-\delta_1(x)}.$$

Observe that we can restrict to $0\leq\Re\sigma<1$, being the negative sigma case symmetrical.

$\diamond$ **Case** $0<\Re\sigma<1$**:** When $x\to 0$, the term $x^{-\sigma}$ is dominant over $\delta_0(x)$ and $\delta_1(x)$. But constant terms and x^{σ} may be of higher order than $\delta_1(x)$ and $\delta_0(x)$. Thus:

$$y(x)=\frac{x\Big(ax^{-\sigma}+O(x^{1-\sigma_1-\Re\sigma})+O(1)+O(x^{\Re\sigma})\Big)}{r_1+O(x^{1-\sigma_1})\ +x\Big(ax^{-\sigma}+O(x^{1-\sigma_1-\Re\sigma})+O(1)+O(x^{\Re\sigma})\Big)}$$

$$=\frac{ax^{1-\sigma}\Big(1+O(\max\{x^{1-\sigma_1},x^{\Re\sigma}\})\Big)}{r_1\Big(1+O(\max\{x^{1-\sigma_1},x^{1-\Re\sigma}\})\Big)}=\frac{a}{r_1}x^{1-\sigma}\ \Big(1+O(\max\{x^{1-\sigma_1},x^{\Re\sigma},x^{1-\Re\sigma}\})\Big)$$

Restoring the value of a, we find the following critical behavior when $0<\Re\sigma<\sigma_1<1$:

$$y(x)=\frac{[\sigma^2-(\theta_0-\theta_x)^2][(\theta_0+\theta_x)^2-\sigma^2]}{16\sigma^3 r}x^{1-\sigma}\ \Big(1+O(\max\{x^{1-\sigma_1},x^{\Re\sigma}\})\Big). \tag{53}$$

$\diamond$ **Case** $\Re\sigma=0$**,** $\sigma\neq 0$**:** In this case $\delta_0(x)$ and $\delta_1(x)$ are $O(x^{1-\sigma_1})$, for any $0<\sigma_1<1$. We can choose σ_1 as small as we like. Also note that $x^{\pm\sigma}=O(1)$, namely it is bounded for $x\to 0$ and does not vanish. Thus:

$$y(x)=\frac{x(ax^{-\sigma}+bx^{\sigma}+c)\ +x\delta_0(x)}{r_1\left[1-\frac{\delta_1(x)}{r_1}+x\Big(O(1)+O(x^{1-\sigma_1})\Big)\right]}=\frac{x}{r_1}\Big(ax^{-\sigma}+bx^{\sigma}+c+r_1\delta_0(x)\Big)\ (1+O(x^{1-\sigma_1}))$$

Now, if we substitute a,b,c and write $x^{\sigma}=\exp\{\sigma\ln x\}$, we obtain:

$$y(x)=x\ \left\{iA\sin\left(i\sigma\ln x+i\ln\frac{2r}{\sigma A}\right)+\frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2}+\delta(x)\right\}(1+\hat{\delta}(x)), \tag{54}$$

where

$$A=\frac{\sqrt{[\sigma^2-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-\sigma^2]}}{2\sigma^2},\qquad \delta(x),\ \hat{\delta}(x),\ \delta^*(x)\ =O(x^{1-\sigma_1}).$$

□

Note that if $\sigma\in\{\pm(\theta_0+\theta_x),\pm(\theta_0-\theta_x)\}$, A is zero, and the coefficient of $x^{-\sigma}$ in $(A_0)_{12}$ becomes zero. Newertheless, $y(x)$ is well defined, staring with power x and $x^{1+\sigma}$ $(-1<\Re\sigma<1)$. It is given by (23), (24). If moreover $\sigma=i\nu$, $\nu\in\mathbf{R}\backslash\{0\}$, $y(x)$ becomes (28), (29).

The leading term extracted in (53) holds for $0\leq\Re\sigma<1$. If instead we choose $-1<\Re\sigma\leq 0$, we would extract the term $x^{1+\sigma}$. Suppose then that, for σ and $\tilde{\sigma}$, with

$0 \le \Re\sigma < 1$ and $-1 < \Re\tilde{\sigma} \le 0$ respectively, we have the two solutions of a given PVI: $y(x) \sim ax^{1-\sigma}$ and $\tilde{y}(x) \sim \tilde{a}x^{1+\tilde{\sigma}}$. Clearly:

$$a = \frac{[\sigma^2 - (\theta_0 - \theta_x)^2][(\theta_0 + \theta_x)^2 - \sigma^2]}{16\sigma^3 r}, \qquad \tilde{a} = -\frac{\tilde{r}}{\tilde{\sigma}}$$

If y and $\tilde{y}$ are the same branch corresponding to the same monodromy data, then $\mathrm{Tr}(M_0 M_x) = 2\cos(\pi\sigma) \equiv 2\cos(\pi\tilde{\sigma})$, namely $\tilde{\sigma} = -\sigma$, and me must have $a = \tilde{a}$. Namely:

$$\tilde{r} = \frac{\sigma^2 A^2}{4\, r} \tag{55}$$

We remark that r, given in (32), does not vanish for the values of $\sigma \in \{\pm(\theta_0+\theta_x), \pm(\theta_0 - \theta_x)\}$. On the other hand, A is vanishes, and so $\tilde{r}$ and the first term $ax^{1-\sigma} \equiv \tilde{a}x^{1+\tilde{\sigma}}$. This is nothing but the fact that the expansion is in this case is (23), (24), and $\tilde{r}$ is not a good integration constant in this case (see also Remark **2)** in section 7).

We estabilish the invariance of (25) when $\sigma \mapsto -\sigma$. Observe that, for purely imaginary σ and $\tilde{\sigma}$, we have $y(x) = x\{iA\sin(i\sigma \ln x + \phi) + B + \delta^*(x)\}$, $\tilde{y}(x) = x\{i\tilde{A}\sin(i\tilde{\sigma}\ln x + \tilde{\phi}) + \tilde{B} + \delta^*(x)\}$, where $\phi = i\ln(2r/\sigma A)$ $\tilde{\phi} = i\ln(2\tilde{r}/\tilde{\sigma}\tilde{A})$. Again, If y and $\tilde{y}$ are the same branch corresponding to the same monodromy data, we have $\tilde{y} = y$, with $\tilde{\sigma} = -\sigma$. Clearly, $\tilde{A} = A$, $\tilde{B} = B$ and the relation (55) implies $\tilde{\phi} = -\phi + (2k+1)\pi$, $k \in \mathbf{Z}$. This means that $\sigma \mapsto -\sigma$ leaves (54) (namely (25)) invariant.

We estabilish the invariance of (16) when $\sigma \mapsto 2-\sigma$. This is done as above, this time observing that the role of σ is played by $1-\sigma$, and $\tilde{r} = (\sigma-1)^2 A^2/4r$, where A is (17) (recall the construction of $y(x)$ by a symmetry transformation in Section 4, and recall that $\phi = i\ln(2r/(1-\sigma)A))$. This implies that $\sigma \mapsto 2-\sigma$ induces $\tilde{\phi} = -\phi + (2k+1)\pi$.

9 Appendix II: Elliptic Representation

In this paper, all the critical behaviors are revised for any σ such that $0 \le \Re\sigma \le 1$ $\sigma \ne 0, 1$. In [15] all the critical behaviors for any $0 \le \Re\sigma \le 1$, $\sigma \ne 0, 1$, are also obtained using the elliptic representation of PVI [6] . If $0 < \Re\sigma < 1$, the behavior (20) is exactly the behavior (60) computed in [15].

But when $\Re\sigma = 0, 1$, the critical behaviors of $y(x)$ obtained in [15], namely (61) and (62) below, are apparently different (27) and (19). Now, (27) must coincide with (61), and (19) with (62). They are just written in a different way. This coincidence allows to prove the convergence of the series of $\delta(x)$ and $\delta^*(x)$.

Before showing this coincidence, let us review the elliptic representation of a Painlevé VI function. This is:

$$y(x) = \wp\left(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x); \omega_1, \omega_2\right) + \frac{1+x}{3}, \quad \nu_1, \nu_2 \in \mathbf{C}, \tag{56}$$

where ω_1, ω_2 are the half-periods. ω_1 is the hypergeometric function:

$$\omega_1(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \tag{57}$$

[6] Actually, for any $\sigma \notin (-\infty, 0] \cup [1, \infty)$. But $\Re\sigma < 0$ or > 1 is equivalent to $0 \le \Re\sigma \le 1$.

and

$$\omega_2(x) = -\frac{i}{2}[F\left(\frac{1}{2},\frac{1}{2},1;x\right)\ln(x) + F_1(x)], \qquad |\arg x| < \pi$$

$$F_1(x) := \sum_{n=0}^{\infty} \frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2} 2\left[\psi(n+\frac{1}{2}) - \psi(n+1)\right] x^n, \tag{58}$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z), \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2, \quad \psi(1) = -\gamma, \quad \psi(a+n) = \psi(a) + \sum_{l=0}^{n-1}\frac{1}{a+l}.$$

The function $v(x)$ solves a non linear equation equivalent to PVI, and in [15] it is proved that it has a convergent expansion. Namely, for any complex ν_1 and ν_2 , such that $\nu_2 \not\in (-\infty, 0] \cup \{1\} \cup [2, +\infty)$, there exists a sufficiently small $\epsilon < 1$ and a solution $v(x)$ such that:

$$v(x) = \sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm} x^n \left[e^{-i\pi\nu_1} x^{1-\nu_2}\right]^m + \sum_{n\geq 0, m\geq 1} c_{nm} x^n \left[e^{i\pi\nu_1} x^{\nu_2}\right]^m$$

$$= \sum_{m_1 m_2} v_{m_1 m_2} x^{m_1(1-\nu_2)+m_2\nu_2}, \quad m_1 + m_2 \geq 1, \quad m_1,\ m_2 \geq 0.$$

a_n, b_{nm}, c_{nm} are certain rational functions of $\alpha, \beta, \gamma, \delta, \nu_2$. The series is proved to converge (see [15]) and defines an holomorphic function of $x, x^{\nu_2}, x^{1-\nu_2}$ in the domain:

$$\mathcal{D} = \{x \in \widetilde{\mathbf{C}\backslash\{0\}} \mid |x| < \epsilon,\ |e^{i\pi\nu_1} x^{\nu_2}| < \epsilon,\ |e^{-i\pi\nu_1} x^{1-\nu_2}| < \epsilon\},$$

$$\mathcal{D} = \{x \in \widetilde{\mathbf{C}\backslash\{0\}} \mid |x| < \epsilon\} \text{ if } \Im\nu_2 = 0.$$

The critical behavior will be determined by the exponent ν_2, which is identified with σ in the following way:

$$\nu_2 = 1 - \sigma \text{ if } \nu_2 \not\in (1,2)$$

$$\nu_2 = 1 + \sigma \text{ if } \nu_2 \in (1,2)$$

where (a, b) is the notation for an open interval.

The asymptotic behavior of $y(x)$ is obtained from the Fourier expansion of the $\wp$-function. Let the "modular parameter" be:

$$\tau(x) = \frac{\omega_2(x)}{\omega_1(x)} = \frac{1}{\pi}\left(\arg x - i\ln\frac{|x|}{16}\right) - \frac{i}{\pi}\left(\frac{F_1}{F} + \ln 16\right)$$

Note that $F_1/F + \ln 16 = O(x)$. The elliptic function can be expanded, when $x \to 0$, as a convergent Fourier series, under the condition (satisfied in $\mathcal{D}$) that:

$$\Im\tau \geq \left|\Im\left(\frac{\nu_1\omega_1 + \nu_2\omega_2 + v}{2\omega_1}\right)\right|.$$

The expansion is:

$$y(x) = \wp(\nu_1\omega_1 + \nu_2\omega_2 + v; \omega_1, \omega_2) + \frac{1+x}{3} =$$

$$\left(\frac{\pi}{2\omega_1}\right)^2 \left\{ -\frac{1}{3} + \sin^{-2}\left(\frac{f}{2}\right) + 8\sum_{n\geq 1} \frac{ne^{2i\pi n\tau}}{1-e^{2i\pi n\tau}} \left[1-\cos\left(nf\right)\right] \right\} + \frac{1+x}{3} \qquad (59)$$

where $f := \nu_1 + \nu_2\tau + \frac{v}{\omega_1}$. Note that in $\mathcal{D}$, $\left|e^{if(x)}\right| < 1$ and $\sin\left(\frac{\pi}{2}f\right) \neq 0$. Namely, the denominator in the expansion does not vanish in $\mathcal{D}$.

Now let us consider the case $0 \leq \Re\sigma \leq 1$, $\sigma \neq 0, 1$, namely $0 \leq \Re\nu_2 \leq 1$, $\nu_2 \neq 0, 1$. In this case,

$$\mathcal{D} = \{x \mid 0 < |x| < \epsilon\}.$$

The other cases (namely, $\Re\sigma < 0$, $\Re\sigma > 1$, $\sigma \notin (-\infty, 0] \cup [1, \infty)$) are equivalent to the above, as it is proved in [15].

9.1 Case $0 < \Re\nu_2 \leq 1$, namely $0 \leq \Re\sigma < 1$

We expand (59) when $x \to 0$, keeping dominant terms:

$$y(x) = \frac{x}{2} - 4e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2} e^{i\frac{\pi v(x)}{\omega_1}} - 4e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2} e^{-i\frac{\pi v(x)}{\omega_1}} +$$

$$+ O(\max\{x\ x^{\nu_2},\ x\ x^{2-\nu_2},\ x^{2\nu_2},\ x^2,\ x^{4-2\nu_2}\})$$

$\diamond$ **Case** $0 < \Re\nu_2 < 1$**, namely** $0 < \Re\sigma < 1$**:** In this case $v(x) \to 0$ for $x \to 0$, $e^{i\pi\frac{v(x)}{\omega_1(x)}} = 1 + O(x) + O(x^{\nu_2}) + O(x^{1-\nu_2})$. From the expansion of $v(x)$ and (59) we compute:

$$y(x) = -4e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}(1+\delta_E(x)), \quad \nu_2 = 1-\sigma. \qquad (60)$$

$$\delta_E(x) = \sum_{m_1\geq 0, m_2\geq 0, m_1+m_2\geq 1} \delta_{m_1m_2} x^{m_1(1-\nu_2)+m_2\nu_2} = O(\max\{x^{\nu_2}, x^{1-\nu_2}\}).$$

$\delta_{m_1m_2} \in \mathbf{C}$. This behavior coincides with (20). The series $\delta_E(x)$ converges in $\mathcal{D}$ and coincides with (21). This proves the convergence of (21).

Remark: For $1 < \nu_2 < 2$, we obtain:

$$y(x) = -4e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}(1+O(\max\{x^{2-\nu_2}, x^{\nu_2-1}\})), \quad \nu_2 = 1+\sigma$$

$\diamond$ **Case** $\Re\nu_2 = 1$**, i.e.** $\Re\sigma = 0$ Now $v(x) \not\to 0$, namely:

$$v(x) = \phi(x) + O(x), \qquad \phi(x) := \sum_{m\geq 1} b_{0m}\left[e^{-i\pi\nu_1}x^{1-\nu_2}\right]^m \not\to 0 \text{ as } x \to 0$$

and $e^{i\pi\frac{v}{\omega_1}} = e^{2i\phi}(1+O(x))$. The series of $\phi(x)$ converges in $\mathcal{D}$. The dominant terms in the Fourier expansion are (note that x, x^{ν_2} and $x^{2-\nu_2}$ are of the same order):

$$y(x) = \frac{x}{2} - 4e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2} e^{i\frac{\pi v(x)}{\omega_1}} - 4e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2} e^{-i\frac{\pi v(x)}{\omega_1}} + O(x^2)$$

Expanding $v(x)$ and (59) we get:

$$y(x) = x\left[\sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\phi(x)\right)\ +\delta_E^*(x)\right],\quad 1-\nu_2=\sigma \tag{61}$$

$$\delta_E^*(x)=\sum_{m_1\geq -1,m_2\geq 1} a_{m_1m_2}x^{m_1(1-\nu_2)+m_2\nu_2}=O(x),\qquad a_{m_1m_2}\in\mathbf{C}.$$

The series converges in $\mathcal{D}$.

9.2 Case $\Re\nu_2=0$, i.e $\Re\sigma=1$

We observe that $v(x)$ does not vanish when $\mathcal{V}=0$, because $x^{\nu_2}\not\to 0$. Namely

$$v(x)=\psi(x)+O(x),\qquad \psi(x):=\sum_{m\geq 1}c_{0m}\left[e^{i\pi\nu_1}x^{\nu_2}\right]^m\not\to 0 \text{ as } x\to 0$$

The series of $\psi(x)$ converges in $\mathcal{D}$. We keep the term $\sin^{-2}(f/2)$ and immediately compute:

$$y(x)=\left[\frac{1}{\sin^2\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)+O(x)\right)}+O(x^2)\right](1+O(x))\ +\frac{x}{2}+O(x^2)$$

$$=\left[\sin^2\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\right)+O(x)\right]^{-1}\ (1+O(x)),\quad \nu_2=1-\sigma.$$

If we perform a more explicit computation from (59) and the expansion of $v(x)$, we get:

$$y(x)=\left[\sin^2\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\right)+\sum_{m_1\geq 1}\sum_{m_2\geq -1}A_{m_1m_2}x^{m_1(1-\nu_2)+m_2\nu_2}\right]^{-1}\ \times$$

$$\times\Big(1+\sum_{m_1\geq 1}\sum_{m_2\geq 0}D_{m_1m_2}x^{m_1(1-\nu_2)+m_2\nu_2}\Big)$$

where $A_{m_1m_2}, B_{m_1m_2}, D_{m_1m_2}\in\mathbf{C}$. The denominator does not vanish on $\mathcal{D}$. The series are convergent in $\mathcal{D}$. We can also apply the the symmetry transformation (30) to (61) and obtain:

$$y(x)=\left\{\sin^2\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\right)+\delta_E^*(x)\right\}^{-1} \tag{62}$$

where

$$\delta_E^*(x)=\sum_{m_1\geq 1,m_2\geq -1}e_{m_1m_2}x^{m_1\sigma+m_2(1-\sigma)}=O(x),\quad e_{m_1m_2}\in\mathbf{C}$$

is a convergent series in $\mathcal{D}$.

9.3 Representation of solution with oscillatory expansions. The bridge between the elliptic representation and the results of this paper

The identification of (27) with (61), and (19) with (62) is done as follows. We rewrite (61) and (62) in terms of new integration constants $\sigma = 1 - \nu_2$ and ϕ_E, instead of ν_2, ν_1 (the substitution is obvious). Thus, (61) is:

$$y_E(x) = x\left[\sin^2\left(i\frac{\sigma}{2}\ln x + \phi_E + \sum_{n\geq 1} c_n(\sigma)[e^{-2i\phi_E}x^{\sigma}]^n\right) + \delta_E^*(x)\right], \tag{63}$$

$$\Re\sigma = 0, \quad |x| < \epsilon, \quad |e^{-2i\phi_E}x^{\sigma}| < \epsilon.$$

and (62) is:

$$y_E(x) = \left[\sin^2\left(i\frac{1-\sigma}{2}\ln x + \phi_E + \sum_{n\geq 1} c_n(\sigma)[e^{-2i\phi_E}x^{1-\sigma}]^n\right) + \delta_E^*(x)\right]^{-1}, \tag{64}$$

$$\Re\sigma = 1, \quad |x| < \epsilon, \quad |e^{-2i\phi_E}x^{1-\sigma}| < \epsilon.$$

On the other hand, we have computed the behaviors:

$$y(x) = x\left[-2iA\sin^2\left(i\frac{\sigma}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) + iA + B + \delta^*(x)\right], \quad \Re\sigma = 0.$$

$$y(x) = x\left[-2iA\sin^2\left(i\frac{1-\sigma}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) + iA + B + \delta^*(x)\right]^{-1}, \quad \Re\sigma = 1.$$

The two results must coincide, beeing associated to the same monodromy data. . The coincidence is explained by the fact that one can always find an oscillatory function $f(x)$ such that:

$$-2iA\sin^2\left(\frac{\nu}{2}\ln x + \frac{\phi}{2} - \frac{\pi}{4}\right) \ + iA + B = \sin^2\left(\frac{\nu}{2}\ln x + f(x)\right), \quad \nu \in \mathbf{R}. \tag{65}$$

If $f(x)$ admits a series expansion (in a suitable domain of convergence), then it must have the following form:

$$f(x) = \sum_{n\geq 0} f_n x^{-i\nu x} \tag{66}$$

This is exactly the form of the functions in the argument of $\sin^2(\ ..\ )$ in (63) and (64) (just write $\sigma = -i\nu$ and $\sigma = 1 + i\nu$ respectively). This proves the convergence of (18) and (26).

The solution $f(x)$ of (65) is constructed as follows. Let $\psi = \frac{\phi}{2} - \frac{\pi}{4}$. (65) becomes the equation:

$$e^{4if} + 2\Big[iAe^{2i\psi} + (2B-1)x^{-i\nu} + iAe^{-2i\psi}x^{-2i\nu}\Big]\ e^{2if} \ + x^{-2i\nu} = 0$$

Let f_1, f_2 be the two solutions:

$$e^{2i(f_1+f_2)} = x^{-2i\nu},$$

$$e^{2if_1} = -iAe^{2i\psi} - (2B-1)x^{-i\nu} - iAe^{-2i\psi}x^{-2i\nu} +$$

$$-iAe^{2i\psi}\sqrt{\left[1+\frac{2B-1}{iA}e^{-2i\psi}x^{-i\nu}+e^{-4i\psi}x^{-2i\nu}\right]^2+\frac{1}{A^2}e^{-4i\psi}x^{-2i\nu}}$$

The square root is such $-\pi < \arg(\sqrt{\ldots}) < \pi$. We observe that e^{2if_1} is clearly an oscillatory function. Further observe that the square root is of the form:

$$\sqrt{1+ae^{-2i\psi}x^{-i\nu}+be^{-4i\psi}x^{-2i\nu}+ce^{-3i\psi}x^{-3i\nu}+de^{-8i\psi}x^{-4i\nu}}$$

where a, b, c, d are constants that can be immediately computed. If the absolute value of the sum of the last four terms is less then 1 we expand the root in series. In particular, this is true if $|e^{-2i\psi}x^{-i\nu}| < r$, for r suitably small. Thus:

$$e^{2if_1} = -iAe^{2i\psi} - (2B-1)x^{-i\nu} - iAe^{-2i\psi}x^{-2i\nu} - iAe^{2i\psi}\Big(1+\sum_{n\geq 1} a_n (e^{-2i\psi}x^{-i\nu})^n\Big)$$

$$f_1 = \psi + \frac{1}{2i}\ln(-2iA) + \frac{1}{2i}\ln\left(1+\frac{2B-1}{2iA}e^{-2i\psi}x^{-i\nu}+\frac{1}{2}e^{-4i\psi}x^{-2i\nu}+\frac{1}{2}\sum_{n\geq 1} a_n(e^{-2i\psi}x^{-i\nu})^n\right)$$

f_1 is an oscillatory function. If in a suitable domain the expansion is possible, we expand the logarithm and obtain:

$$f_1(x) = \psi + \frac{1}{2i}\ln(-2iA) \ + \sum_{n\geq 1} b_n (e^{-2i\psi}x^{-i\nu})^n$$

$$f_2(x) = -f_1 - \nu \ln x$$

Note that the last formula implies:

$$\sin^2\Big(\frac{\nu}{2}\ln x + f_2\Big) = \sin^2\Big(\frac{\nu}{2}\ln x + f_1\Big).$$

□

9.4 Example of Picard solutions

Picard [28] studied the case $\theta_0 = \theta_x = \theta_1 = 0$, $\theta_\infty = 1$. This section is written in order to show the general results realized in an example that can be computed in terms of classical special functions (elliptic and hypergeometric). In this case the function appearing in the elliptic representation is $v(x) = 0$. Thus:

$$y(x) = \wp\left(\nu_1\omega_1(x)+\nu_2\omega_2(x);\omega_1,\omega_2\right) + \frac{1+x}{3}, \quad \nu_1,\ \nu_2 \in \mathbf{C},$$

Apply the Fourier expansion to:

$$\wp\left(\nu_1\omega_1(x)+\nu_2\omega_2(x);\omega_1,\omega_2\right) = \wp\left(\nu_1\omega_1(x)+[\nu_2+2N]\omega_2(x);\omega_1,\omega_2\right)$$

The domain of convergence is:

$$\left|\Im\left[\frac{\nu_1}{2}+\left(\frac{\nu_2}{2}+N\right)\tau(x)\right]\right|<\Im\tau(x)$$

Namely:

$$(\Re\nu_2+2+2N)\ln\frac{|x|}{16}+O(x)<\Im\nu_2\ \arg x+\pi\ \Im\nu_1<(\Re\nu_2-2+2N)\ln\frac{|x|}{16}+O(x)\quad(67)$$

This is larger than $\mathcal{D}$. The critical behavior for $x\to 0$ is computed along the paths:

$$\arg x=\arg x_0+\frac{\Re\nu_2+2N-\mathcal{V}}{\Im\nu_2}\ln|x|,\qquad -2\le\mathcal{V}\le 2,\quad \Im\nu_2\neq 0$$

If $\Im\nu_2=0$ we take a radial path.

The critical behavior is then obtained by extracting the leading terms of the Fourier expansion. We do this straightforwardly if $0\le\mathcal{V}<2$. The other cases are obtained from the previous one by changing $N\mapsto N\pm 1$. Results:

$\diamond$ For $0<\mathcal{V}<1$

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]x^{\nu_2+2N}\ \left(1+O(x^{\nu_2+2N},x^{1-\nu_2-2N})\right)$$

$\diamond$ For $1<\mathcal{V}<2$

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]^{-1}x^{2-\nu_2-2N}\ \left(1+O(x^{2-\nu_2-2N},x^{\nu_2+2N-2})\right)$$

$\diamond$ For $\mathcal{V}=1$

$$y(x)=x\ \left[\sin^2\left(i\frac{1-\nu_2-2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}\right)\ +O(x)\right]$$

$\diamond$ For $\mathcal{V}=0$

$$y(x)=\left[\frac{1}{\sin^2\left(-i\frac{\nu_2+2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}-i\frac{\nu_2+2N}{2}\left[\frac{F_1(x)}{F(x)}+\ln 16\right]\right)}\ +O(x^2)\right]\times$$

$$\times\left(1-\frac{x}{2}+O(x^2)\right)+\ \frac{x}{2}+O(x^2)$$

Namely:

$$y(x)=\sin^{-2}\left(-i\frac{\nu_2+2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}-i\frac{\nu_2+2N}{2}\left[\frac{F_1(x)}{F(x)}+\ln 16\right]\right)\ (1+O(x))\ +O(x)$$

$$=\left[\sin^2\left(-i\frac{\nu_2+2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}\right)\ +O(x)\right]^{-1}(1+O(x))+O(x)$$

$\diamond$ For $\mathcal{V}=2$:

$$y(x)=\left[\sin^2\left(i\frac{2-\nu_2-2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}\right)\ +O(x)\right]^{-1}(1+O(x))+O(x)$$

$\diamond$ For $-1<\mathcal{V}<0$: behavior of case $1<\mathcal{V}<2$ with $N\mapsto N+1$.

$\diamond$ For $-2<\mathcal{V}<-1$: behavior of case $0<\mathcal{V}<1$ with $N\mapsto N+1$.

$\diamond$ For $\mathcal{V}=-1$: behavior of case $\mathcal{V}=1$ with $N\mapsto N+1$.

$\diamond$ For $\mathcal{V}=-2$: behavior of case $\mathcal{V}=0$ with $N\mapsto N+1$.

$\diamond$ If $\Im\nu_2=0$, we choose the convention $0\leq\nu_i<2$. The critical behavior for $0<\nu_2<1$ is the same of the case $\Im\nu_2\neq 0$ with $N=0$ and $0<\mathcal{V}<1$; for $1<\nu_2<2$ it is the same of the case $\Im\nu_2\neq 0$ with $N=0$ and $1<\mathcal{V}<2$. Finally, in special cases we have Taylor expansions:

$$y(x)=x\ \left[\sin^2\left(\frac{\pi\nu_1}{2}\right)+\sum_{n\geq 1}a_nx^n\right],\qquad \text{if } \nu_2=1$$

$$y(x)=\sin^{-2}\left(\frac{\pi\nu_1}{2}\right)+\sum_{n\geq 1}a_nx^n,\qquad \text{if } \nu_2=0,\quad \nu_1\neq 0$$

Observe that the choice of N is arbitrary, therefore the *same* transcendent has different critical behaviors on different domains (67) specified by different values of N.

Remark: Note that in the cases $\mathcal{V}=-2,0,2$, the denominator $\sin^2(...)$ may vanish in the domain (67). Therefore, there may be movable poles. The position of the poles can be determined if we keep $F_1(x)/F(x)$ in the argument of $\sin^2(\ ..\ )$ and set $\sin^2(\ ..\ )=0$.

Now let $N=0$ and $\nu_2=i\nu$, $\nu\in\mathbf{R}$. Identify $\sigma=1-\nu_2$. When $\mathcal{V}=0$, $x\to 0$ along a radial path $\arg x=$ constant. The behavior becomes:

$$y(x)=\sin^{-2}\left(\frac{\nu}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\frac{\nu}{2}\left[\frac{F_1(x)}{F(x)}+\ln 16\right]\right)\ (1+O(x))\ +O(x)\tag{68}$$

Let $N=0$ and $\nu_2=1+i\nu$, $\nu\in\mathbf{R}$, and $\sigma=1-\nu_2$. When $\mathcal{V}=1$, $x\to 0$ along a radial path, and the behavior becomes:

$$y(x)=\ x\ \left[\sin^2\left(\frac{\nu}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\frac{\nu}{2}\right)\ +O(x)\right]$$

From the above computations, we see that the critical behavior of the Picard solutions is in accordance with our general results.

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